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3rd Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: Tue, November 17th, 2015

Exercise 1 (properties of the tangential derivative)
Let $\Gamma \subseteq \mathbb{R}^{n+1}$ and $W \subseteq \mathbb{R}^{n+1}$ an open neighborhood of Γ.

a) Prove the product rule for $f, g \in C^1(W, \mathbb{R})$:

$$\nabla_\Gamma(fg) = f\nabla_\Gamma g + g\nabla_\Gamma f.$$

b) For $f \in C^1(W, \mathbb{R}^m)$ the m-dimensional tangential derivative is given by

$$D_\Gamma f := \begin{pmatrix} \nabla_\Gamma f_1 & \cdots & \nabla_\Gamma f_m \end{pmatrix}^T \in \mathbb{R}^{m \times n}.$$

Show that $D_\Gamma f = Df\,(I - \nu\nu^T)$ where Df is the usual Jacobian of f.

c) Derive a chain rule for D_Γ for $f \in C^1(W, \mathbb{R}^m)$ and $g \in C^1(\mathbb{R}^m, \mathbb{R})$.

Exercise 2 (properties of the oriented distance function)
Consider the oriented distance function $d\colon \mathbb{R}^{n+1} \to \mathbb{R}$ for an oriented hypersurface Γ. Prove:

a) The function d is globally Lipschitz and there exists a $\delta \in \mathbb{R}_{>0}$ such that $d \in C^2(\Gamma_\delta)$ for $\Gamma_\delta := \left\{x \in \mathbb{R}^{n+1} \mid |d(x)| < \delta\right\}$.

b) There exists $\delta \in \mathbb{R}_{>0}$ such that for all $x \in \Gamma_\delta$ exists a unique $a(x) \in \Gamma$ with $x = a(x) + d(x)\nu(a(x))$.

c) We have $\|\nabla d(x)\| = 1$ for all $x \in \Gamma$.

Remark: Here (and in any other exercise, unless stated otherwise) you may assume that the hypersurface Γ is compact.

Exercise 3 (symmetry of the curvature matrix)
Let $\Gamma \subseteq \mathbb{R}^{n+1}$ an orientable hypersurface. Show that the curvature matrix $(H_{jk})_{j,k\in\{1,\ldots,n+1\}}$, which is defined by

$$\forall j, k \in \{1, \ldots, n+1\}\colon\ H_{jk} := \underline{D}_j \nu_k,$$

is symmetric.

Please turn over...

Exercise 4 (intrinsic tangent space)
Let $\Gamma \subseteq \mathbb{R}^{n+1}$ a C^1-hypersurface and $V \subseteq \Gamma$. Furthermore, let $U \subseteq \mathbb{R}^n$ open, $x_0 \in U$ and $\Phi \in C^1(U, V)$ a C^1-diffeomorphism.
Define $y_0 := \Phi(x_0)$. Show that the intrinsic tangent space

$$\hat{T}_{y_0}\Gamma := \left\{T \in \mathbb{R}^{n+1} \mid \exists \varepsilon \in \mathbb{R}_{>0}\ \exists \gamma \in C^1((-\varepsilon, \varepsilon), U):\ \gamma(0) = x_0\ \wedge\ D(\Phi \circ \gamma)(0) = T\right\}$$

coincides with the tangent space $T_{y_0}\Gamma$ in the sense of the lecture.

Have fun!