Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Tobias Kies

5th Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: ***presumably*** **Wed, December 9th, 2015**
Please note the announcements on the website regarding organizational information on the lecture and the tutorial in the upcoming weeks.

Exercise 1 (representation of orthogonal projections)
Let $n \in \mathbb{N}$, $k \in \{0, \dots, n\}$ and $V \subseteq \mathbb{R}^n$ a k-dimensional linear subspace. For any $x \in \mathbb{R}^n$ the (euclidean) orthogonal projection of x onto V is defined to be the solution of

$$\min_{y \in V} \|y - x\|, \qquad (*)$$

denoted by Px.
Now, suppose $a_1, \dots, a_k \in \mathbb{R}^n$ form a basis of V. Show that $Px = A(A^T A)^{-1} A^T x$ for all $x \in \mathbb{R}^{n+1}$ where $A = (a_1, \dots, a_k) \in \mathbb{R}^{n \times k}$.

Remark: You may use without proof that $(*)$ admits a unique solution.

Exercise 2 (derivative and curvature vector's representation in parameterization)
Let $\Gamma \subseteq \mathbb{R}^{n+1}$ a C^2-hypersurface. We consider a local parameterization of Γ by a map $X \in C^2(V, \mathbb{R}^{n+1}$ with $V \subseteq \mathbb{R}^n$ open.
The Riemannian metric is denoted by $G := (g_{ij})_{i,j \in \{1,\dots,n\}} = \left(\frac{\partial X}{\partial \theta_i} \cdot \frac{\partial X}{\partial \theta_j}\right)_{i,j \in \{1,\dots,n\}}$ and its inverse by $(g^{ij})_{i,j \in \{1,\dots,n\}}$. Furthermore, we define $g := \det(G)$.

a) For $f \in C^1(U, \mathbb{R})$, where U is an open neighborhood of $X(V)$, show that

$$\nabla_\Gamma f \circ X = \sum_{i,j=1}^{n} g^{ij} \frac{\partial (f \circ X)}{\partial \theta_j} \frac{\partial X}{\partial \theta_i}.$$

b) For the case $n = 1$, show that

$$(H \circ X)(\nu \circ X) = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial \theta_i} \left(g^{ij} \sqrt{g} \frac{\partial X}{\partial \theta_j} \right).$$

Bonus: Prove this formula for arbitrary $n \in \mathbb{N}_{>0}$.

Have fun!