Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Tobias Kies

6th Exercise for the Lecture

Numerics IV

WiSe 2015/2016
http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: Tue, December 15th, 2015

Exercise 1 (shape derivative of area elements)
Let Γ_0 an orientable hypersurface and $X\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ a sufficiently smooth transport mapping with $X(0,\cdot) = \mathrm{id}_{\mathbb{R}^{n+1}}$. Furthermore, define $\Gamma(t) := X(t,\Gamma_0)$ for all $t \in [0,T]$ and let $V(t,\cdot) = \frac{\partial}{\partial t} X(t,\cdot)$ the velocity field associated to the transport X.
Denote $\gamma\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R},\ (t,x) \mapsto \det(DX(t,x))$ the volume area element and $\omega\colon [0,T] \times \Gamma_0 \to \mathbb{R},\ (t,x) \mapsto \gamma(t,x)\|DX(t,x)^{-T} \cdot \nu(x)\|$ the surface area element where ν is the outer unit normal on Γ_0.
Assuming that $\Gamma(t)$ is an orientable hypersurface for all $t \in [0,T]$, show that

a)
$$\frac{\partial}{\partial t}\gamma(t)\bigg|_{t=0} = \nabla \cdot V(0)$$

b)
$$\frac{\partial}{\partial t}\omega(t)\bigg|_{t=0} = \nabla_\Gamma \cdot V(0)$$

Exercise 2 (shape derivative of volume area functional)
Let $\Omega \subseteq \mathbb{R}^{n+1}$ compact and $X \in C^1\left([0,T] \times \Omega, \mathbb{R}^{n+1}\right)$ such that $\det(DX(t,\cdot)) > 0$ for all $(t,x) \in [0,T] \times \Omega$ and $X(0,\cdot) = \mathrm{id}_\Omega$. Define $\Omega(t) := X(t,\Omega)$ for $t \in [0,T]$.
Compute the shape derivative of the volume area functional, i. e. compute

$$J'(\Omega)(\partial_t X) := \lim_{t \searrow 0} \frac{1}{t}\left(J(\Omega(t)) - J(\Omega(0))\right)$$

where we define for all (bounded and measurable) $A \subseteq \mathbb{R}^{n+1}$

$$J(A) := \int_A 1 \mathrm{d}x.$$

Show that $J'(\Omega)(\partial_t X)$ only depends on $\partial_t X|_{\partial\Omega}$.

Have fun!