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7th Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: Tue, January 5th, 2015

Exercise 1 (integration by parts on surfaces)
Let $\Gamma \subseteq \mathbb{R}^{n+1}$ a closed, compact C^2-hypersurface and $f \in C^1(\mathbb{R}^{n+1})$, $g \in C^2(\mathbb{R}^{n+1})$, $\phi \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Prove that

a) $\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g \, \mathrm{d}\Gamma = -\int_\Gamma f \, \Delta_\Gamma g \, \mathrm{d}\Gamma$

b) $\int_\Gamma H\nu \cdot \phi \, \mathrm{d}\Gamma = \int_\Gamma \nabla_\Gamma \, \mathrm{id}_{\mathbb{R}^{n+1}} \cdot \nabla_\Gamma \phi \, \mathrm{d}\Gamma.$

Remark: You may use without proof that for all $f \in C^1(\mathbb{R}^{n+1})$

$$\forall i \in \{1, \dots, n+1\} \colon \quad \int_\Gamma \underline{D}_i f \, \mathrm{d}\Gamma = \int_\Gamma f H \nu_i \, \mathrm{d}\Gamma.$$

Exercise 2 (gradient flows: Ginzburg-Landau energy)
Let $\varepsilon \in \mathbb{R}_{>0}$, $\Omega \subseteq \mathbb{R}^n$ compact and

$$\begin{aligned} J \colon H^1(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega \frac{\varepsilon}{2} \|\nabla u(x)\|^2 + \frac{1}{2\varepsilon} \left(u(x)^2 - 1\right)^2 \, \mathrm{d}^n x. \end{aligned}$$

Given $u_0 \in H^1(\Omega)$, derive a $L^2(\Omega)$-gradient flow $(u(t))_{t \in [0,T]}$ with respect to J and $u(0) = u_0$.

Bonus: Let $H = H_0^1(\Omega)$ and $\langle u, v \rangle_H := \langle \nabla \Delta^{-1} u, \nabla \Delta^{-1} v \rangle_{L^2(\Omega)}$ where $w = \Delta^{-1} f$ iff $\Delta w = f$ (in the weak sense) and $w|_{\partial\Omega} = 0$. For $u_0 \in H_0^1(\Omega)$, derive the corresponding H-gradient flow of J.

Please turn over...

Exercise 3 (shape derivative for constant integrand)
Let $\Omega \subseteq \mathbb{R}^{n+1}$ compact and $X \in C^1\left([0,T] \times \Omega, \mathbb{R}^{n+1}\right)$ such that $\det(DX(t,\cdot)) > 0$ for all $(t,x) \in [0,T] \times \Omega$ and $X(0,\cdot) = \mathrm{id}_\Omega$. Define $\Omega(t) := X(t,\Omega)$ for $t \in [0,T]$.
Given $Y \in C^1(\mathbb{R}^{n+1})$ compute the shape derivative $J'(\Omega(0))(\partial_t X)$ where

$$J(A) := \int_A Y(x)\mathrm{d}x$$

for any compact measurable set $A \subseteq \mathbb{R}^{n+1}$.

Have fun!