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8th Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

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Exercise 1 (discretization of curve shortening flow)
Given $I := [0, 2\pi]$ and $X_0 \colon I \to \mathbb{R}^2$ the parametric curve shortening flow equation for $X \colon I \times [0,T) \to \mathbb{R}^2$ is given by

$$\begin{aligned} X_t - \frac{1}{|X_\theta|}\left(\frac{X_\theta}{|X_\theta|}\right)_\theta &= 0 && \text{on } I \times (0,T) \\ X(\cdot, 0) &= X_0 && \text{on } I \\ X(0,\cdot) &= X(2\pi, \cdot) && \text{on } [0,T). \end{aligned}$$

Let $N \in \mathbb{N}_{>0}$, $h = 2\pi/N$, $\theta_j := jh$ $j \in \{0,\dots,N\}$ and $I_j = [\theta_{j-1}, \theta_j]$ for $j \in \{1,\dots,N\}$. Furthermore, choose the finite element space

$$S_h := \left\{ v \in C(I, \mathbb{R}^2) \mid v|_{I_j} \text{ linear for all } j \in \{1,\dots,n\},\, v(0) = v(2\pi) \right\}.$$

Then the corresponding spatial discretization is given by

$$\forall v \in S_h\colon \quad \int_I X_{h,t} |X_{h,\theta}|\, v \,\mathrm{d}(x) + \int_I X_{h,t} |X_{h,\theta}|^{-1}\, v \,\mathrm{d}(x) = 0.$$

a) Use the basis representation $X_h(\theta,t) = \sum_{j=0}^n X_j(t)\lambda_j(\theta)$ where the $\lambda_j \colon I \to \mathbb{R}$, $j \in \{0,\dots,N\}$ are the nodal piecewise linear basis functions on the grid $(\theta_0,\dots,\theta_N)$. Show that above discretization then is given by the following system of ODEs:

$$\forall j \in \{1,\dots,N\}\colon \ \frac{1}{6} q_j \dot{X}_{j-1} + \frac{1}{3}(q_j + q_{j+1})\dot{X}_j + \frac{1}{6} q_{j+1}\dot{X}_{j+1} = \frac{X_{j+1} - X_j}{q_{j+1}} - \frac{X_j - X_{j-1}}{q_j}$$

where $X_j = X_{j+N}$ for $j \in \{-1, 0, 1\}$ and $q_j = |X_j - X_{j-1}|$.

b) Inexact integration (mass lumping) can lead to the simplified system:

$$\forall j \in \{1,\dots,N\}\colon \ \frac{1}{2}(q_j + q_{j+1})\dot{X}_j = \frac{X_{j+1} - X_j}{q_{j+1}} - \frac{X_j - X_{j-1}}{q_j}.$$

Which quadrature rule has been used here?

Write down the implicit/explicit Euler schemes for solving this system of ODEs.

Please turn over...

Exercise 2 (gradient flow on discrete curve length)
Let $X_h(\theta) = \sum_{j=1}^N X_j\lambda_j(\theta) \in S_h$ with vectors $X_j \in \mathbb{R}^2$ be the basis representation of a linear finite element approximation of a curve parameterization $X\colon I \to \mathbb{R}^2$. Assume for simplicity that S_h is defined on an equidistant grid.
Derive a discretization of the curve shortening flow by imposing a discrete gradient flow on the curve length functional with respect to the X_j, i. e. differentiate

$$\int_I \|X_{h,\theta}\| \,\mathrm{d}(x)$$

with respect to the vectors X_j and derive a system $X_{h,t} = -DX_h$. Compare the resulting discretization with other known discretizations.

Have fun!