Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Tobias Kies

9th Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: Tue, January 19th, 2015

Exercise 1 (some gradient transformations)
Let $(\theta_0, \dots, \theta_N)$ and $I = \bigcup_{j=1}^{N} I_j$ where $I_j := [\theta_{j-1}, \theta_j]$. Furthermore, let $X_h \colon I \to \mathbb{R}^2$ such that $X|_{[\theta_{j-1}, \theta j]}$ is affine linear for all $j \in \{1, \dots, N\}$. Define $\Gamma_h := X_h(I)$ and

$$S_h := \left\{ v \in C(\Gamma_h, \mathbb{R}^2) \mid v|_{X(I_j)} \text{ is affine linear for all } j \in \{1, \dots, N\} \right\}.$$

Then show that:

a) $\forall f \in C^1(\Gamma_h)\colon \quad (\nabla_{\Gamma_h} f) \circ X_h = \dfrac{(f \circ X_h)_\theta X_{h,\theta}}{\|X_{h,\theta}\|^2}$

b) $\forall v \in S_h\colon \quad (\nabla_{\Gamma_h} v) \circ X_h = \dfrac{(v \circ X_h)_\theta \otimes X_{h,\theta}}{\|X_{h,\theta}\|^2}$

Exercise 2 (non-coercivity of surface energy)
Let $\Omega \subseteq \mathbb{R}^n$. Prove or disprove that the function

$$\begin{aligned} J\colon H^1(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega \sqrt{1 + \|\nabla u(x)\|^2}\, \mathrm{d}x \end{aligned}$$

is coercive in $H^1_0(\Omega)$.

Remark: In this context, for a Banach space $(V, \|\cdot\|_V)$ and a function $f\colon V \to V'$ we say that f is coercive if and only if

$$\lim_{\|u\|_V \to \infty} \frac{f(u)(u)}{\|u\|_V} = +\infty$$

holds.

Please turn over...

Exercise 3 (mean curvature flow with tangent motion)
Consider $X, Y\colon I \times [0,T] \to \mathbb{R}^2$. Let X and Y fulfill

$$X_t = \frac{1}{|X_\theta|}\left(\frac{X_\theta}{|X_\theta|}\right)_\theta, \qquad Y_t = \frac{Y_{\theta\theta}}{|Y_\theta|^2}.$$

Show that X_t and Y_t have the same evolution in normal direction. In particular, Y defines a curve that evolves in normal direction like a mean curvature flow.

Exercise 4 (surface decrease of mean curvature flow)
Let $\Omega \subseteq \mathbb{R}^n$ and $u\colon \Omega \times [0,T] \to$ (sufficiently) smooth. Show that if $u(t,\cdot) \in H_0^1(\Omega)$ for all $t \in [0,1]$ and if u satisfies

$$\forall t \in (0,T)\ \forall v \in H_0^1(\Omega)\colon \quad \int_\Omega \frac{u_t(t,x)v(x) + \nabla_x u(t,x)\cdot\nabla v(x)}{\sqrt{1+\|\nabla_x u(t,x)\|^2}}\,\mathrm{d}x = 0$$

then we have

$$\forall t \in (0,T)\colon \quad \int_\Omega \frac{u_t(t,x)^2}{\sqrt{1+\|\nabla_x u(t,x)\|^2}}\,\mathrm{d}x + \frac{\partial}{\partial t}|\Gamma(t)\| = 0$$

where $\Gamma(t)$ is the surface described by $u(t,\Omega)$ and where $|\Gamma(t)|$ is its surface area.

Have fun!