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10th Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: Tue, January 26th, 2015

Exercise 1 (surface decrease for mean curvature flow)
Let $\Omega \subseteq \mathbb{R}^n$ and $u \colon \Omega \times [0,T] \to$ (sufficiently) smooth. Show that if $u(t,\cdot) \in H_0^1(\Omega)$ for all $t \in [0,1]$ and if u satisfies

$$\forall t \in (0,T)\ \forall v \in H_0^1(\Omega)\colon\ \int_\Omega \frac{u_t(t,x)v(x) + \nabla_x u(t,x) \cdot \nabla v(x)}{\sqrt{1 + \|\nabla_x u(t,x)\|^2}}\,\mathrm{d}x = 0$$

then we have

$$\forall t \in (0,T)\colon\ \int_\Omega \frac{u_t(t,x)^2}{\sqrt{1 + \|\nabla_x u(t,x)\|^2}}\,\mathrm{d}x + \frac{\partial}{\partial t}|\Gamma(t)| = 0$$

where $\Gamma(t)$ is the surface described by $u(t,\Omega)$ and where $|\Gamma(t)|$ is its surface area.

Exercise 2 (estimates on time derivatives for some geometric quantities)
Let $\Omega \subseteq \mathbb{R}^n$ and $Q(u) := \sqrt{1 + \|\nabla u\|^2}$, $\nu(u) := \frac{1}{Q(u)}\left(\nabla u, -1\right)^T$ for all $u \in H^1(\Omega)$.
Show for $u \in H^1((0,T), H^2(\Omega))$:

a) $\frac{\partial}{\partial t} Q(u(\cdot)) \leq \|\nabla u_t\|$ and

b) $\left\|\frac{\partial}{\partial t}\nu(u(\cdot))\right\| Q(u) \leq 2\|\nabla u_t\|$.

Exercise 3 (discrete approximation of normal vector)
Let $\Omega \subseteq \mathbb{R}^n$ and $Q(u) := \sqrt{1 + \|\nabla u\|^2}$, $\nu(u) := \frac{1}{Q(u)}\left(\nabla u, -1\right)^T$ for all $u \in H^1(\Omega)$.
Furthermore, let $S \subseteq H^2(\Omega)$, $h \in \mathbb{R}_{>0}$ and $I := H^2(\Omega) \to S$ such that

$$\exists c \in \mathbb{R}_{>0}\, \forall u \in H^2(\Omega)\, \forall k \in \{0,1,2\}\colon\ \|u - Iu\|_{H^k(\Omega)} \leq ch^{2-k}\|u\|_{H^2(\Omega)}.$$

Prove that there exists $c \in \mathbb{R}_{>0}$ such that

$$\forall u \in H^2(\Omega)\colon\ \int_\Omega \|\nu(u) - \nu(Iu)\|^2 Q(Iu) \leq ch^2.$$

Please turn over...

Exercise 4 (a version of Grönwall's inequality)
Let $\alpha \in C([0,T])$, $\beta \in C^1([0,T])$ and $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\forall t \in [0,T] \colon \; \alpha(t) + \beta'(t) \leq c_1 \beta(t) + c_2.$$

Show the following version of the Grönwall's inequality for $t \in [0,T]$:

$$\int_0^t \alpha(s) \, \mathrm{d}s + \beta(t) \leq e^{c_1 t} \left(\frac{c_2}{c_1} + \beta(0) \right).$$

Have fun!