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11th Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

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Exercise 1 (discrete mean curvature stability)
Let $S_h \subseteq H^1(\Omega)$ a discretization of $H^1(\Omega)$, $\tau \in \mathbb{R}_{>0}$, $u_h^0 \in S_h$ and $u_h^{m+1} \in S_h$ the solution of

$$\forall v \in S_h\colon\ \frac{1}{\tau}\int_\Omega \frac{u_h^{m+1} v}{Q(u_h^m)} + \int_\Omega \frac{\nabla u_h^{m+1}\cdot\nabla v}{Q(u_h^m)} = \frac{1}{\tau}\int_\Omega \frac{u_h^m v}{Q(u_h^m)}$$

for all $m \in \{0,\dots,M-1\}$ where $Q(u_h^m) = \sqrt{1+\|\nabla u_h^m\|^2}$. Prove that

$$\tau \sum_{k=0}^{m-1} \int_\Omega |V_h^k|^2 Q(u_h^k) + \int_\Omega Q(u_h^m) \le \int_\Omega Q(u_h^0)$$

holds for all $m \in \{1,\dots,M\}$ where $V_h^k = -\frac{u_h^{k+1}-u_h^k}{\tau Q(u_h^k)}$.

Exercise 2 (fully discrete mean curvature error estimate)
Let $u\colon [0,T]\times\Omega \to \mathbb{R}^2$ sufficiently smooth with $\partial_t u, \nabla u \in L^\infty([0,T]\times\Omega)$ and $u_h^m \in S_h$ for $m \in \{0,\dots,M\}$ where $h \in \mathbb{R}_{>0}$ and $S_h \subseteq H^1(\Omega)$ is some discretization of $H^1(\Omega)$. Furthermore, let $\tau := \frac{T}{M}$ and define $u^m := u(m\tau,\cdot)$, $\nu^m = \frac{1}{Q(u^m)}(\nabla u^m, -1)^T$, $\nu_h^m = \frac{1}{Q(u_h^m)}(\nabla u_h^m, -1)^T$ for $m \in \{0,\dots,M\}$ and $V^m := -\frac{\partial_t u^m}{Q(u^m)}$, $V_h^m := -\frac{u_h^{m+1}-u_h^m}{\tau Q(u_h^m)}$ for $m \in \{0,\dots,M-1\}$ where $Q(v) := \sqrt{1+\|\nabla v\|^2}$ for $v \in H^1(\Omega)$. Finally, assume that $\|Q(u_h^m)\|_{L^\infty(\Omega)}$ is bounded independently of τ and h and

$$\tau \sum_{m=0}^{M-1} \int_\Omega |V^m - V_h^m|^2 Q_h^m + \sup_{m\in\{0,\dots,M\}} \int_\Omega \|\nu^m - \nu_h^m\|^2 Q_h^m \le c(\tau^2+h^2)$$

holds for some $c \in \mathbb{R}_{>0}$. Prove that there exists $\hat{c} \in \mathbb{R}_{>0}$ independent of τ and h such that

$$\tau \sum_{m=0}^{M-1} \int_\Omega \left|\partial_t u^m - \frac{u_h^{m+1}-u_h^m}{\tau}\right|^2 + \sup_{m\in\{0,\dots,M\}} \int_\Omega \|\nabla u^m - \nabla u_h^m\|^2 \le \hat{c}(\tau^2+h^2).$$

Have fun!