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1st exercise for the lecture

Numerics IV

Winter Term 2016/2017

http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Tuesday, Nov 1st, 2016

Exercise 1 (8 TP)
Let H be a Hilbert space and $f\colon H \to \mathbb{R}$ Gâteaux-differentiable. The function f is called strongly convex with modulus $\mu > 0$ if for all $x, y \in H$ and $\lambda \in (0,1)$

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) - \lambda(1-\lambda)\frac{\mu}{2}\|x-y\|^2.$$

a) Show that f is convex if and only if

$$f(x) - f(y) \ge \langle Df(y), x-y \rangle \qquad \forall x, y \in H. \tag{1}$$

b) Show that f is convex if and only if $Df\colon H \to H'$ is monotone, i. e.,

$$\langle Df(x) - Df(y), x-y \rangle \ge 0 \qquad \forall x, y \in H. \tag{2}$$

c) Show that f is strongly convex with modulus $\mu > 0$ with if and only if

$$f(x) - f(y) \ge \langle Df(y), x-y \rangle + \frac{\mu}{2}\|x-y\|^2 \qquad \forall x, y \in H. \tag{3}$$

d) Show that f is strongly convex with modulus $\mu > 0$ if and only if $Df\colon H \to H'$ is strongly monotone, i. e.,

$$\langle Df(x) - Df(y), x-y \rangle \ge \mu\|x-y\|^2 \qquad \forall x, y \in H. \tag{4}$$

Please turn over...

Exercise 2 (4 TP)
Let $\underline{\psi}, \overline{\psi} \in H^1(\Omega)$ with $\underline{\psi} \leq \overline{\psi}$ (almost everywhere). Show that the set

$$K = \{v \in H^1(\Omega) \,|\, \underline{\psi} \leq v \leq \overline{\psi} \text{ a. e.}\}$$

is convex and closed in $H^1(\Omega)$ and that its characteristic function χ_K is convex, proper, and lower semicontinuous. Furthermore show that for $\underline{\psi} \neq \overline{\psi}$ the set K is unbounded and χ_K is not coercive.

Exercise 3 (4 TP)
Find a Hilbert space H and a function $f\colon H \to \mathbb{R}$ such that f is coercive over all one-dimensional subspaces of H but not over the full space H.

Remark: Let $(X, \|\cdot\|)$ a normed vector space and $f\colon X \to \mathbb{R} \cup \{\infty\}$.

- We call f *proper* if there exists $x_0 \in X$ such that $f(x_0) \neq \infty$.
- The function f is *coercive* if $\lim_{\|x\| \to \infty} f(x) = \infty$, or more precisely

$$\forall c \in \mathbb{R}_{>0} \; \exists r \in \mathbb{R}_{>0}\colon \; \|x\| \geq r \implies f(x) \geq c.$$

- We say that f is *lower semicontinuous* (l. s. c.) if for all $x \in X$ the inequality $\liminf_{z \to x} f(z) \geq f(x)$ is true, or more precisely: For all $x \in X$ and all sequences $(x_k)_{k \in \mathbb{N}}$ in X holds

$$\lim_{k \to \infty} \|x_k - x\| = 0 \implies \liminf_{k \to \infty} f(x_k) \geq f(x).$$

- Let $K \subseteq X$ a set. Then the *characteristic function* χ_K is here defined as

$$\chi_K\colon X \longrightarrow \mathbb{R} \cup \{\infty\}$$
$$x \longmapsto \begin{cases} 0 & \text{for } x \in K \\ \infty & \text{for } x \notin K. \end{cases}$$

Have fun!