Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser, Tobias Kies

2nd exercise for the lecture

Numerics IV

Winter Term 2016/2017

http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Tuesday, Nov 8th, 2016

Exercise 1 (6 TP)
Let V a reflexive Banach-space, $J_0\colon V \to \mathbb{R}$ convex and Gâteaux-differentiable, $\varphi\colon V \to \mathbb{R}$ convex and $K \subset V$ convex. Show that the minimization problem

$$u \in K : \qquad J(u) \leq J(v) \qquad \forall v \in K$$

for $J = J_0 + \varphi$ is equivalent to the variational inequality

$$u \in K : \qquad \langle J_0'(u), v - u \rangle + \varphi(v) - \varphi(u) \geq 0 \qquad \forall v \in K.$$

What happens if $J = \frac{1}{2}a(\cdot,\cdot) - l(\cdot)$ is a quadratic functional?
Hint: Consider $\tilde{v} = (1-\lambda)u + \lambda v$. Show for convex $f \in C^1(\mathbb{R})$

$$f(y) - f(x) \geq f'(x)(y - x) \qquad \forall x, y \in \mathbb{R}.$$

Exercise 2 (3 TP)
Let $H := L^2([0,1])$. Find a sequence $(x_k)_{k\in\mathbb{N}} \subseteq H$ and $x \in H$ such that $x_k \rightharpoonup x$ in H but $x_k \not\to x$ in H for $k \to \infty$.

Exercise 3 (3 TP)
Let V a vector space, $X \subseteq V$ a convex set and $f\colon X \to \mathbb{R}$ convex. Given $n \in \mathbb{N}$ points $x_i \in X$ and weights $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^n \lambda_i = 1$, prove the inequality

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

Please turn over...

Exercise 4 (2 TP)
Let X a Banach space with dual space X'. Suppose sequences $(x_k)_{k\in\mathbb{N}}$ in X and $(x'_k)_{k\in\mathbb{N}}$ in X' as well as elements $x \in X$ and $x' \in X'$. Show that if $x_k \rightharpoonup x$ in X and $x'_k \to x'$ in X' then $x'_k(x_k) \to x'(x)$ in $\mathbb{R}$ for $k \to \infty$.

Exercise 5 (2 TP)
Let H a Hilbert space, $x \in H$ and $(x_k)_{k\in\mathbb{N}}$ a sequence in H. Prove

$$x_k \to x \text{ in } H \text{ for } k \to \infty \iff x_k \rightharpoonup x \text{ in } H \text{ and } \|x_k\| \to \|x\| \text{ in } \mathbb{R} \text{ for } k \to \infty.$$

Have fun!