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3rd exercise for the lecture

Numerics IV

Winter Term 2016/2017

`http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php`

Due: Tuesday, Nov 15th, 2016

Exercise 1 (8 TP + 2 Bonus TP)
Let X a real, reflexive Banach space and $A\colon X \to X'$ monotone and coercive, i. e.

$$\forall u, v \in X\colon\ \langle A(u) - A(v), u - v\rangle \geq 0 \tag{1}$$

and

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to \infty \text{ for } \|u\|_X \to \infty. \tag{2}$$

Furthermore suppose that A is hemicontinuous, i. e. the map

$$t \longmapsto \langle A(u + tv), w\rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in X$. We consider the equation

$$Au = b \text{ in X'} \tag{3}$$

for some $b \in X'$. The set of solutions to (3) is denoted by S.

a) Let $u \in X$. Prove that

$$\forall v \in X\colon\ \langle b - Av, u - v\rangle \geq 0 \implies Au = b.$$

b) Show that the set S is bounded in X.

c) Show that S is convex.

d) Show that S is closed in X.

e) Assume that A is strictly monotone, i. e. for $u, v \in X$ with $u \neq v$ holds ">" instead of "$\geq$" in (1). Prove that then at most one solution to (3) can exist.

Please turn over...

Exercise 2 (8 TP + 2 Bonus TP)
Let $p \in (1, \infty)$ and $\Omega \subseteq \mathbb{R}^d$ bounded with smooth boundary. We want to solve for $p' := \frac{p}{p-1}$ and $f \in L^{p'}(\Omega)$ the quasilinear elliptic PDE

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= f \text{ on } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

with corresponding weak formulation

$$\forall v \in H_0^{1,p}(\Omega)\colon \quad \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x.$$

To this end we define $X := H_0^{1,p}(\Omega)$ and

$$\begin{aligned} A(u)v := \langle A(u), v\rangle &:= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, \mathrm{d}x \\ b(v) := \langle b, v\rangle &:= \int_\Omega f v \, \mathrm{d}x. \end{aligned}$$

a) Show that the operator $A\colon X \to X'$ is well-defined and bounded and that b is bounded, i. e. prove $A(u) \in X'$ and $\|A(u)\|_{X'} < \infty$ for all $u \in X$ and $\|b\|_{X'} < \infty$.

b) Let $J(u) := \int_\Omega \|\nabla u\|^p \, \mathrm{d}x$ for all $u \in X$. Show that J is Gâteaux-differentiable with $DJ(u)v = A(u)v$ for all $u, v \in X$.

c) Show that A is a monotone operator, i. e.

$$\forall u, v \in X\colon \ \langle A(u) - A(v), u - v\rangle \geq 0.$$

Bonus: Show that A is strictly monotone, i. e. that "$>$" holds for $u \neq v$.

d) Show that A is coercive, i. e. that for all sequences $(u_k)_{k\in\mathbb{N}}$ with $\|u_k\|_X \to \infty$ for $k \to \infty$ also follows $\frac{\langle Au, u\rangle}{\|u\|_X} \to \infty$ for $k \to \infty$.

Hint: Remember the Hölder-inequality and that the *dual norm on* X' is defined by

$$\forall x' \in X'\colon \quad \sup_{x \in X\setminus\{0\}} \frac{|x'(x)|}{\|x\|_X}.$$

Have fun!