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4th exercise for the lecture

Numerics IV

Winter Term 2016/2017

http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Tuesday, Nov 22nd, 2016

Exercise 1 (4 TP)
Let $\Omega \subseteq \mathbb{R}^d$ and $\underline{\psi}, \overline{\psi} \in L^2(\Omega)$ such that $\underline{\psi} \leq \overline{\psi}$ almost everywhere and define

$$K := \left\{ v \in L^2(\Omega) \mid \underline{\psi} \leq v \leq \overline{\psi} \text{ a. e.} \right\}.$$

For $f \in L^2(\Omega)$ we consider the projection problem

$$\min_{u \in K} \|u - f\|^2_{L^2(\Omega)}. \tag{1}$$

a) Show that (1) admits a unique solution.

b) Compute the solution of (1) explicitly.

Exercise 2 (2 TP)
Let $A \in \mathbb{R}^{n \times n}$ symmetric positive definite and $a_i, b_i \in \mathbb{R}$ such that $a_i \leq b_i$ for $i \in \{1, \dots, n\}$. Define $K := \prod_{i=1}^n [a_i, b_i]$, suppose $f \in \mathbb{R}^n$, $u \in K$ and assume the variational inequality

$$\forall v \in K\colon\ \langle Au, v - u \rangle \geq \langle f, v - u \rangle$$

is fulfilled. Show that for all $i \in \{1, \dots, n\}$ with $u_i \in (a_i, b_i)$ the equality $(Au)_i = f_i$ holds.

Exercise 3 (2 TP)
Let X a Banach space, $K \subseteq X$ convex and $f\colon K \to \mathbb{R} \cup \{\infty\}$ convex. Show that the subdifferential of f is monotone, i. e. prove

$$\forall x, y \in K\ \forall u \in \partial f(x), v \in \partial f(y)\colon \langle u - v, x - y \rangle \geq 0.$$

Please turn over...

Exercise 4 (4 TP)
Let X a normed vector space, $K \subseteq X$ nonempty, convex and closed and suppose an elliptic bilinear form $a\colon X \times X \to \mathbb{R}$. Define the operator induced by a by

$$A\colon X \longrightarrow X', \qquad x \longmapsto \begin{pmatrix} X \to \mathbb{R} \\ y \mapsto a(x,y) \end{pmatrix}$$

and denote I_K the characteristic function over K, i. e. we have for all $x \in X$

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{if } x \notin K. \end{cases}$$

Suppose $b \in X'$ and that $x \in X$ fulfills $(A + \partial I_K)x \ni b$. Show that x is then the solution to the minimization problem

$$\min_{y \in K} \frac{1}{2}\langle A(y), y\rangle - b(y).$$

Exercise 5 (4 TP)
Let X a normed vector space and $J\colon X \to \mathbb{R} \cup \{\infty\}$ convex. The *polar function of* J is defined as

$$J^*\colon X' \longrightarrow \mathbb{R} \cup \{\infty\}, \qquad x' \longmapsto \sup_{x \in X} \big(x'(x) - J(x)\big).$$

Verify that J^* is convex, prove

$$\partial J(x) = \big\{x' \in X' \mid J^*(x') = x'(x) - J(x)\big\}$$

and show that for all $x \in X$ and $x' \in X'$ holds

$$x \in (\partial J)^{-1}(x') \implies Ex \in (\partial J^*)(x')$$

where $Ex \in X''$ is defined by $(Ex)(y') = y'(x)$ for all $y' \in X'$.

Remark: For a Banach space X and $f\colon X \supseteq K \to \mathbb{R} \cup \{\infty\}$ convex the *subdifferential of f in $x \in K$* is given by

$$\partial f(x) := \big\{v \in X' \mid \forall y \in K\colon\ f(x) + \langle v, y - x\rangle \le f(y)\big\}$$

and its inverse in $x' \in X'$ by

$$(\partial f)^{-1}(x') := \big\{x \in X \mid x' \in \partial f(x)\big\}.$$