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5th exercise for the lecture

Numerics IV

Winter Term 2016/2017

http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Tuesday, Nov 29th, 2016

Exercise 1 (4 TP)
Let $\Omega \subseteq \mathbb{R}^n$. Prove or disprove that the function

$$\begin{aligned} J\colon H^1(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega \sqrt{1 + \|\nabla u(x)\|^2}\,\mathrm{d}x \end{aligned}$$

is coercive in $H_0^1(\Omega)$.

Exercise 2 (4 TP)
Let $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ convex and I_h the interpolation into the first order FE-space S_h on a simplex-triangulation of Ω. Then the lumped approximation

$$\varphi_h(v) = \int_\Omega I_h(\Phi \circ v)(x)$$

of

$$\varphi(v) = \int_\Omega \Phi(v(x))$$

is stable in the sense that $\varphi(v) \leq \varphi_h(v)$ for all $v \in S_h$.

Please turn over...

Exercise 3 (4 TP)
Let $\Omega \subseteq \mathbb{R}^d$ a bounded open domain with smooth boundary, let $T \in \mathbb{R}_{>0}$ and suppose that $u\colon \Omega \times [0,T],\ (x,t) \mapsto u(x,t)$ is sufficiently smooth and integrable and solves the PDE

$$\begin{aligned} \partial_t u &= \sqrt{1+\|\nabla u\|^2}\ \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) \quad \text{on } \Omega \times (0,T) \\ u &= 0 \text{ on } \partial\Omega \times (0,T) \end{aligned}$$

Show that

$$\frac{\partial}{\partial t} \int_\Omega \sqrt{1+\|\nabla u(x,t)\|^2}\, \mathrm{d}x \leq 0.$$

Exercise 4 (4 TP)
Let $\Omega \subseteq \mathbb{R}^d$ a bounded domain with smooth boundary, let $f \in L^2(\Omega)$, let $\underline{\psi}, \overline{\psi} \in H_0^1(\Omega)$ such that $\underline{\psi} \leq \overline{\psi}$ almost everywhere and define

$$K := \left\{ v \in H_0^1(\Omega) \mid \underline{\psi} \leq v \leq \overline{\psi} \text{ a. e.} \right\}.$$

Suppose $u \in H_0^1(\Omega)$ is a solution of the obstacle problem

$$\min_{v \in K} \int_\Omega \sqrt{1+\|\nabla v(x)\|^2}\, \mathrm{d}x - \int_\Omega f(x)v(x)\, \mathrm{d}x.$$

Define the *inactive set* as

$$\mathcal{I} := \left\{ x \in \Omega \mid u(x) > \underline{\psi}(x) \ \wedge\ u(x) < \overline{\psi}(x) \right\}$$

and denote its interior as $\mathcal{I}^\circ$. Show that if $u|_{\mathcal{I}}$ is sufficiently smooth then

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) = f \text{ on } \mathcal{I}^\circ$$

holds almost everywhere.

Have fun!