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5th exercise for the lecture

Numerics IV

Winter Term 2016/2017

http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Tuesday, Dec 6th, 2016

Exercise 1 (4 TP)
Let $(H, \langle\cdot,\cdot\rangle_H)$ and $(G, \langle\cdot,\cdot\rangle_G)$ be real Hilbert spaces with $H \subseteq G$. Suppose that we are given a function $J\colon H \to \mathbb{R}$ that is Gâteaux-differentiable on H. We say that the G-gradient of J in $u \in H$ exists if there is an element $g \in G$ such that

$$\forall v \in H\colon\ \ DJ(u)(v) = \langle g, v\rangle_G$$

where $DJ(u) \in H'$ is the Gâteaux-derivative of J in u. In this setting we define $\nabla_G J(u) := g$.

a) Show that $\nabla_H J(u)$ exists for all $u \in H$.

b) Give an example for spaces H, G and a function $J\colon H \to \mathbb{R}$ such that J is Gâteaux-differentiable but the G-gradient of J does not exist for all $u \in H$.

c) Assume that $\nabla_G J(u)$ exists. Show that $-\nabla_G J(u)$ is the direction of steepest descent for J in u with respect to the G-metric, i. e. prove that $-\nabla_G J(u)/\|\nabla_G J(u)\|_G$ is the unique minimizer of

$$\min_{g\in H, \|g\|_G=1} DJ(u)(g).$$

Exercise 2 (4 TP)
Let H and G be real Hilbert spaces with $H \subseteq G$ and let $J\colon H \to \mathbb{R}$ be Gâteaux-differentiable. A family $(u(t))_{t\in\mathbb{R}_{\geq 0}} \subseteq H$ is a solution of the G-gradient flow of J with initial condition $u_0 \in H$ if $u(0) = u_0$ and for all $t \in \mathbb{R}_{>0}$

$$\partial_t u(t) = -\nabla_G J(u(t)).$$

Here $\partial_t u(t) := \lim_{h\searrow 0} \frac{1}{h}(u(t+h) - u(t)) \in H$ where the convergence holds with respect to the H-norm.
Show that for all $t \in \mathbb{R}_{>0}$

$$\frac{\partial}{\partial t} J(u(t)) \leq 0$$

and that for J convex

$$\nabla_G J(u(t)) = 0 \iff u(t) \in \operatorname{argmin}_{v\in H} J(v).$$

Hint: You may use without proof that the chain rule holds for Gâteaux-differentiation.

Exercise 3 (4 TP)
Let $H = H_0^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ bounded with smooth boundary and define

$$J\colon H_0^1(\Omega) \to \mathbb{R}, \qquad v \mapsto \frac{1}{2}\int_\Omega \varepsilon\|\nabla u(x)\|^2 + \frac{1}{\varepsilon}\Psi(u(x))\,\mathrm{d}x$$

where $\varepsilon \in \mathbb{R}_{>0}$ and $\Psi\colon \mathbb{R} \to \mathbb{R}$ such that $\Psi \circ v \in L^1(\Omega)$ for all $v \in H_0^1(\Omega)$. Suppose $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and assume that $\Psi \in C^1(\mathbb{R})$ with $\Psi' \circ u \in L^2(\Omega)$. Compute the $L^2(\Omega)$-gradient of J in u. How does this relate to the Allen-Cahn equation?

Exercise 4 (4 TP)
Let Ω be a bounded polyhedral domain, $\mathcal{T}$ a conforming triangulation of Ω by simplices, $\mathcal{S}_\mathcal{T}$ the space of piecewise linear finite element functions on $\mathcal{T}$ and $I_\mathcal{T} : C(\Omega) \to \mathcal{S}_\mathcal{T}$ the nodal interpolation operator. For a convex, proper, l. s. c. functional $\Phi\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ define

$$\varphi : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}, \qquad \varphi(v) = \int_\Omega \Phi(v(x))\,dx.$$

a) Show that φ is proper and convex.

b) Show that the lumped approximation

$$\varphi_\mathcal{T} : \mathcal{S}_\mathcal{T} \to \mathbb{R} \cup \{\infty\}, \qquad \varphi_\mathcal{T}(v) = \int_\Omega I_\mathcal{T}(\Phi \circ v)(x)\,dx$$

is proper and convex and satisfies $\varphi(v) \leq \varphi_\mathcal{T}(v)$ for all $v \in \mathcal{S}_\mathcal{T}$.