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9th exercise for the lecture

Numerics IV

Winter Term 2016/2017

http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Exercise 1: Tuesday, Jan 17th, 2017
Exercise 2: Tuesday, Jan 10th, 2017

Exercise 1 (4 TP)
Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ a symmetric, continuous and coercive bilinear form, $\ell\colon H^1(\Omega) \to \mathbb{R}$ linear and continuous, and $\psi \in C^0(\Omega)$. We consider the obstacle problem

$$\min_{u \in K} \ \frac{1}{2} a(u,u) - \ell(u)$$

where

$$K := \left\{ v \in H_0^1(\Omega) \mid v \geq \psi \right\}.$$

Suppose that $S_h \subseteq H_0^1(\Omega)$ and $K_h \subseteq H^1(\Omega)$ are closed subspaces and that we approximate the solution of the obstacle problem by the solution of

$$\min_{u_h \in K_h} \ \frac{1}{2} a(u_h,u_h) - \ell(u_h).$$

Furthermore, assume that there exists $g \in L^2(\Omega)$ such that for all $v \in H^1(\Omega)$

$$a(u,v) - \ell(v) = \langle g, v \rangle_{L^2(\Omega)}.$$

Show that both problems admit a unique solution, denoted by u and u_h respectively, and that the error estimate

$$c\|u-u_h\|^2_{H^1(\Omega)} \leq \left(\inf_{v_h \in K_h} \|u - v_h\|^2_{H^1(\Omega)} + \|u - v_h\|_{L^2(\Omega)} \right) + \inf_{v \in K} \|u_h - v\|_{L^2(\Omega)}$$

holds with the constant c depending on a and g.

Hint: You may want to state the variational inequalities corresponding to the given minimization problems and to try a strategy that is similar to the proof of Céa's lemma.

Please turn over...

Exercise 2 (4TP + 4 Bonus TP)
Let $\Omega = [-1, 1]$, and $c_f, c_\psi \in \mathbb{R}$, and define $f(x) = c_f$, $\psi(x) = c_\psi$ for all $x \in \Omega$. Determine for which pairs (c_f, c_ψ) the obstacle problem

$$\min_{u \in K} \int_\Omega \|\nabla u(x)\|^2 \,\mathrm{d}x - \int_\Omega f(x)\, u(x) \,\mathrm{d}x$$

with

$$K := \left\{ v \in H_0^1(\Omega) \mid v \geq \psi \right\}$$

admits a solution and compute an explicit expression for the solution (in case it exists). When computing this expression you may assume without proof that there exists a (possibly empty) interval $I = [-a, a]$ such that $u = \psi$ on I and $u \neq \psi$ on $\Omega \setminus I$.

Hint: Try to prove $u'(a) = 0$. Using this, the boundary data and the knowledge what u looks like on I and $\Omega \setminus I$, you can derive a (small) finite-dimensional system of equations that you can solve in order to obtain an analytical expression for u.

Have fun!