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10th exercise for the lecture

Numerics IV

Winter Term 2016/2017

`http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php`

Due: Theoretical Exercises: Tuesday, Jan 17th, 2017
Programming Exercise: Tuesday, Jan 24th, 2017

Exercise 1 (4 TP, from last week's assignment)
Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ a symmetric, continuous and coercive bilinear form, $\ell\colon H^1(\Omega) \to \mathbb{R}$ linear and continuous, and $\psi \in C^0(\Omega)$. We consider the obstacle problem

$$\min_{u \in K} \frac{1}{2} a(u,u) - \ell(u)$$

where

$$K := \left\{ v \in H_0^1(\Omega) \mid v \geq \psi \right\}.$$

Suppose that $S_h \subseteq H_0^1(\Omega)$ is a closed linear subspace and $K_h \subseteq H^1(\Omega)$ is a closed non-empty convex set and that we approximate the solution of the obstacle problem by the solution of

$$\min_{u_h \in K_h} \frac{1}{2} a(u_h, u_h) - \ell(u_h).$$

Furthermore, assume that there exists $g \in L^2(\Omega)$ such that for all $v \in H^1(\Omega)$

$$a(u,v) - \ell(v) = \langle g, v \rangle_{L^2(\Omega)}.$$

Show that both problems admit a unique solution, denoted by u and u_h respectively, and that the error estimate

$$c\|u - u_h\|_{H^1(\Omega)}^2 \leq \left(\inf_{v_h \in K_h} \|u - v_h\|_{H^1(\Omega)}^2 + \|u - v_h\|_{L^2(\Omega)} \right) + \inf_{v \in K} \|u_h - v\|_{L^2(\Omega)}$$

holds with the constant c depending on a and g.

Hint: You may want to state the variational inequalities corresponding to the given minimization problems and to try a strategy that is similar to the proof of Céa's lemma.

Please turn over...

Exercise 2 (8 TP)
Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $K := \prod_{i=1}^n [\alpha_i, \beta_i] \subseteq \mathbb{R}^n$ where $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i \leq \beta_i$. We define the energy functional

$$J\colon \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad u \longmapsto \frac{1}{2}u^T A u - b^T u + I_K(u)$$

with the characteristic function I_K defined by

$$I_K(u) = \begin{cases} 0 & \text{for } u \in K \\ \infty & \text{for } u \notin K. \end{cases}$$

We are interested in minimizers of J. For $u^0 \in \mathbb{R}^n$, the iterates of the projected Gauß-Seidel method are for $k \in \mathbb{N}$ defined by

$$u^{k+1} := w^{k,n}$$

where $w^{k,0} = u^k$ and

$$w^{k,i} = w^{k,i-1} + e_i \cdot \arg\min_{\lambda\in\mathbb{R}} J(w^{k,i-1} + \lambda e_i)$$

for $i \in \{1, \dots, n\}$ with $e_i := (\delta_{ij})_{j=1,\dots,n} \in \mathbb{R}^n$.
In the following we will use the splitting $A = L + D + R$, where L contains the entries of A below the diagonal, R the entries above the diagonal and D is the diagonal.

a) Show that

$$\arg\min_{\lambda\in\mathbb{R}} J(w^{k,i-1} + \lambda e_i) = \max\left(\alpha_i - u_i^k, \min\left(\beta_i - u_i^k, \frac{(b - Aw^{k,i-1})^T e_i}{e_i^T A e_i}\right)\right).$$

b) Show that the projected Gauß-Seidel method is equivalent to the iteration $\widetilde{u}^{k+1} = \widetilde{u}^k + v^k$ where v^k solves the variational inequality

$$\forall v \in K - \widetilde{u}^k\colon\ \langle (L+D)v^k, v - v^k\rangle \geq \langle b - A\widetilde{u}^k, v - v^k\rangle$$

with $K - \widetilde{u}^k := \{v \in \mathbb{R}^n | v + \widetilde{u}^k \in K\}$.

c) Show that the projected Gauß-Seidel method is equivalent to the iteration $\widetilde{u}^k$ where $\widetilde{u}^{k+1}$ solves the variational inequality

$$\forall v \in K\colon \langle (L+D)u^{k+1}, v - \widetilde{u}^{k+1}\rangle \geq \langle b - R\widetilde{u}^k, v - \widetilde{u}^{k+1}\rangle.$$

d) Show that the projected Gauß-Seidel method is equivalent to the iteration

$$\widetilde{u}^{k+1} = (L + D + \partial I_K)^{-1}(b - R\widetilde{u}^k).$$

e) Show that the projected Gauß-Seidel method is equivalent to the iteration

$$\widetilde{u}^{k+1} = \widetilde{u}^k + v^k, \qquad v^k = (L + D + \partial I_{K-\widetilde{u}^k})^{-1}(b - A\widetilde{u}^k).$$

Exercise 3 (8 PP)
For $A \in \mathbb{R}^{n\times n}$ s.p.d., $b, \underline{\psi}, \overline{\psi} \in \mathbb{R}^n$ with $\underline{\psi} \leq \overline{\psi}$, and $K = \{v \in \mathbb{R}^n \,|\, \underline{\psi} \leq \overline{\psi}\}$ we consider the minimization problem

$$u = \arg\min_{v\in K} \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle.$$

a) Implement a projected Gauß–Seidel increment operator as

```
y = pgs_inc(A,r,lower,upper)
```

computing $y = (D + L + \partial\chi_K)^{-1}r$ for a $D + L + R = A$ splitting of A and `lower` and `upper` representing $\underline{\psi}$ and $\overline{\psi}$, respectively.

b) Implement a projected Gauß–Seidel-step as

```
xnew = pgs(A,b,lower,upper,xold)
```

computing $x_{new} = (D + L + \partial\chi_K)^{-1}(b - Rx_{old})$. Base you implementation on `pgs_inc`.

c) Implement a projected Gauß–Seidel solver as

```
x = pgs_solver(A,b,lower,upper,x0,maxnu)
```

doing `maxnu` Gauß–Seidel iterations returning the last iterate as `x`.

d) Test your method using the test suite provided on the lecture homepage.

e) Illustrate numerically (using a $1d$ example) why the projected Gauß–Seidel method can still be viewed as a smoother for the obstacle problem for the Laplacian.

f) Let $\Omega = B_1(0) \subseteq \mathbb{R}^2$, $f \equiv -4$, $\psi(x) = -\|x\|^2 - \frac{2}{5}$ and

$$K := \{v \in H^1_0(\Omega) \mid v \geq \psi\}.$$

Discretize the problem

$$\min_{u\in K} \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \langle f, u\rangle_{L^2(\Omega)}$$

using piecewise linear finite elements and solve the resulting discrete minimization problem with the projected Gauß-Seidel method. Compare your results with the exact solution

$$u(x) = \begin{cases} \|x\|^2 - 4a^2\ln(\|x\|) - 2a^2 - 0.4 + 4a^2\ln(a) & \text{if } \|x\| \geq a \\ \psi(x) & \text{if } \|x\| < a. \end{cases}$$

where $a \approx 0.29534584846812719$.

Take care to have optimal complexity for sparse matrices in all implementations.

Have fun!