Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser, Tobias Kies

11th exercise for the lecture

Numerics IV

Winter Term 2016/2017

`http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php`

Due: Theoretical Exercises: Tuesday, Jan 24th, 2017
Programming Exercise: Tuesday, Jan 31st, 2017

Exercise 1 (4 TP)

a) Let $\underline{\psi}, \overline{\psi} \in \mathbb{R}^n$ and $K := \{v \in \mathbb{R}^n \mid \underline{\psi} \leq v \leq \overline{\psi}\}$ where the "$\leq$" have to be understood component-wise. Show that the Euclidean projection onto K, denoted by $P\colon \mathbb{R}^n \to K$, is given by

$$P_K(v) = \left(\max\left(\underline{\psi}_i, \min\left(\overline{\psi}_i, v_i\right)\right)\right)_{i \in I} \in \mathbb{R}^n$$

with $I := \{1, \ldots, n\}$.

b) Furthermore, suppose $A \in \mathbb{R}^{n \times n}$ symmetric positive definite and $b \in \mathbb{R}^n$. Define for $v \in \mathbb{R}^n$

$$J_0(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle, \qquad J(v) = J_0(v) + I_K(v).$$

Here I_K is the indicator function over the set K. Prove for $u \in K$ and $v \in \mathbb{R}^n \setminus \{0\}$

$$\operatorname{argmin}_{\rho \in \mathbb{R}} \; J(u + \rho v) = \max\left(\underline{\delta}, \min\left(\overline{\delta}, \frac{\langle b - Au, v\rangle}{\langle Av, v\rangle}\right)\right)$$

where

$$\underline{\delta} = \max\left(\max_{i \in I,\ v_i > 0} \frac{\underline{\psi}_i - u_i}{v_i}, \max_{i \in I,\ v_i < 0} \frac{\overline{\psi}_i - u_i}{v_i}\right)$$
$$\overline{\delta} = \min\left(\min_{i \in I,\ v_i < 0} \frac{\underline{\psi}_i - u_i}{v_i}, \min_{i \in I,\ v_i > 0} \frac{\overline{\psi}_i - u_i}{v_i}\right).$$

Exercise 2 (6 TP)
Let $A \in \mathbb{R}^{n\times n}$ be s. p. d. and $b, \psi \in \mathbb{R}^n$. For the minimization problem

$$u \in K: \qquad J(u) \le J(v) \qquad \forall v \in K$$

with $J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle$ and $K = \{v \in \mathbb{R}^n \mid \psi \le v\}$ and the lower/diagonal/upper splitting $A = L + D + R$ we consider the projected Gauß–Seidel method given by

$$u^{\nu+1} = (D + L + \partial\chi_K)^{-1}(b - Ru^\nu) = u^\nu + F(u^\nu).$$

a) Show that u solves the minimization problem if and only if $F(u) = 0$ for F as defined above.

b) For $v \in \mathbb{R}^n$ we denote by $\mathcal{A}(v) = \{i \in \mid \psi_i = v_i\}$ and by $\mathcal{I}(v) = \{1, \dots, n\} \setminus \mathcal{A}(v)$ the sets of active and inactive indices, respectively. Now let $\mathcal{J} \subset \{1, \dots, n\}$ be any index set and $U(\mathcal{J}) = \{v \in \mathbb{R}^n \mid \mathcal{I}(v + F(v)) = \mathcal{J}\}$, i. e., for all v in $U(\mathcal{J})$ the active set is the same after application of one Gauß–Seidel step. Show that F is affine linear on $U(\mathcal{J})$.

Exercise 3 (4 PP)

a) Implement a function `h1_semi_norm` that computes the error in the $H^1(\Omega)$-seminorm of a function u and a finite element function u_h by approximating $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ using quadrature. See the template file from the provided resources for more details.

b) Let $\Omega = B_1(0) \subseteq \mathbb{R}^2$, $f(x) = -4$, $\underline{\psi}(x) = -\|x\|^2 - \frac{2}{5}$, $\overline{\psi}(x) = 1$ and $K := \{v \in H_0^1(\Omega) \mid \underline{\psi} \le v \le \overline{\psi}\}$. Discretize the problem

$$\min_{v\in K} \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \langle f, v\rangle_{L^2(\Omega)}$$

using piecewise finite linear elements and the grid from the file `grid_unit_circle.mat` in the resources and solve the resulting equation using the projected Gauß-Seidel method. Approximate the error $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ using `h1_semi_norm` and the exact solution

$$u(x) = \begin{cases} \|x\|^2 - 4a^2\ln(\|x\|) - 2a^2 - 0.4 + 4a^2\ln(a) & \text{if } \|x\| \ge a \\ \underline{\psi}(x) & \text{if } \|x\| < a \end{cases}$$

with $a \approx 0.29534584846812719$. Compute the error for various grid refinements and explain your results. What do you observe and why?

Have fun!