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12th exercise for the lecture

Numerics IV

Winter Term 2016/2017

`http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php`

Due: Tuesday, Jan 31st, 2017

Exercise 1 (4 TP)
For $M \in \mathbb{R}^{m\times n}$ the Moore–Penrose pseudoinverse $M^+ \in \mathbb{R}^{n\times m}$ is the matrix which is uniquely determined by the properties

$$MM^+M = M, \qquad M^+MM^+ = M^+, \qquad (MM^+)^T = MM^+, \qquad (M^+M)^T = M^+M.$$

For $M \in \mathbb{R}^{n\times n}$ and an index set $\mathcal{I} \subset \{1,\dots,n\}$ we denote by $M_{\mathcal{I}} \in \mathbb{R}^{n\times n}$ the matrix with $(M_{\mathcal{I}})_{i,j} = M_{i,j}$ for $i,j \in \mathcal{I}$ and $(M_{\mathcal{I}})_{i,j} = 0$ else. Now let $A \in \mathbb{R}^{n\times n}$ and $\mathcal{I} \subset \{1,\dots,n\}$.

a) Show that $A_{\mathcal{I}} = I_{\mathcal{I}} A I_{\mathcal{I}}$

b) Assume that the $|\mathcal{I}| \times |\mathcal{I}|$ submatrix of A indexed by $\mathcal{I}$ is regular. Show that $(A_{\mathcal{I}})^+ = (A_{\mathcal{I}} + I - I_{\mathcal{I}})^{-1} - I + I_{\mathcal{I}}$.

Exercise 2 (4 TP)
Let $A \in \mathbb{R}^{m\times n}$.

a) Suppose A is surjective. Show that $A^+ = A^T(AA^T)^{-1}$.

b) Suppose A is injective. Show that $A^+ = (A^TA)^{-1}A^T$.

c) Suppose A is diagonal. Show that $(A^+)_{ij} = \frac{1}{a_{ij}}$ if $a_{ij} \neq 0$ and $(A^+)_{ij} = 0$ if $a_{ij} = 0$.

d) Suppose A is symmetric with eigenvalue decomposition $A = QDQ^T$ where D is diagonal and Q is orthonormal. Prove $A^+ = QD^+Q^T$.

Please turn over...

Exercise 3 (4 TP)
Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $B \in \mathbb{R}^{m\times n}$.

a) Show that there exists $\lambda_0 \in \mathbb{R}_{>0}$ such that $A+\lambda I$ is invertible for $\lambda \in (-\lambda_0, \lambda_0)\setminus\{0\} =: \Lambda$.

b) Prove that the limits

$$P := \lim_{\Lambda\ni\lambda\to 0} (A+\lambda I)^{-1}A = \lim_{\Lambda\ni\lambda\to 0} A(A+\lambda I)^{-1}$$

exist and coincide and that P is the Euclidean projection onto $\mathrm{im}(A) := \{Ax \mid x \in \mathbb{R}^n\}$.

Hint: Use the orthogonal decomposition $\mathbb{R}^n = \mathrm{im}(A) + \ker(A)$ and the eigenvalue decomposition of A.

c) Show that the limits

$$\overline{B} := \lim_{\lambda\to 0}(B^TB + \lambda^2 I_n)^{-1}B^T = \lim_{\lambda\to 0} B^T(BB^T + \lambda^2 I_m)^{-1}$$

exist and coincide and that $\overline{B}B$ is the Euclidean projection onto $\mathrm{im}(B^TB)$.

Hint: Use the orthogonal decomposition $\mathbb{R}^m = \mathrm{im}(B) + \ker(B^T)$.

Remark: From this it is possible to show that $\overline{B} = B^+$ holds. Thus above limits give an alternative characterization of the Moore-Penrose pseudoinverse.

Exercise 4 (4 TP)
Let $M \in \mathbb{R}^{n\times n}$ symmetric positive definite and let $M^{(k)} \in \mathbb{R}^{n\times n}$ be a sequence of symmetric positive definite matrices such that $M^{(k)} \to M$ for $k \to \infty$. Suppose $\mathcal{A} \subseteq \{1,\dots,n\}$ and let $\mathcal{I} := \{1,\dots,n\}\setminus\mathcal{A}$. Prove that for any sequence $(\alpha_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ with $\alpha_k \to \infty$ for $k \to \infty$ holds

$$\lim_{k\to\infty}(M^{(k)} + \alpha_k I_{\mathcal{A}})^{-1} = (M_{\mathcal{I}})^+.$$

Here $(I_{\mathcal{A}})_{ij} = \delta_{ij}$ for $i \in \mathcal{A}$ and $(I_{\mathcal{A}})_{ij} = 0$ for $i \in \mathcal{I}$ is the reduced identity matrix that only contains the entries associated to the index set $\mathcal{A}$, and similary $(M_{\mathcal{I}})_{ij} = M_{ij}$ for $i,j \in \mathcal{I}$ and $(M_{\mathcal{I}})_{ij} = 0$ else.

Hint: You can use without proof that for a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$ with $A \in \mathbb{R}^{n\times n}$ invertible its *Schur complement* is given by $S = D - CA^{-1}B$, and that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I_n & -A^{-1}B \\ 0 & I_m \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}\begin{pmatrix} I_n & 0 \\ -CA^{-1} & I_m \end{pmatrix}.$$

Use the Schur complement applied to a suitable splitting of the above matrices in order to compute the limit of the inverses.

Have fun!