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13th exercise for the lecture

Numerics IV

Winter Term 2016/2017
http://numerik.mi.fu-berlin.de/wiki/WS_2016/NumericsIV.php

Due: Tuesday, Feb 7th, 2017

Exercise 1 (12 TP)
Let $A \in \mathbb{R}^{n\times n}$ symmetric positive definite, $b \in \mathbb{R}^n$ and $K := \prod_{i=1}^n [\alpha_i, \beta_i]$. We consider the quadratic obstacle problem induced by $J = J_0 + \varphi$ with

$$J_0(x) = \frac{1}{2}x^T Ax - b^T x, \qquad \varphi(x) = \sum_{i=1}^n \varphi(x_i), \qquad \varphi(x_i) = \chi_{[\alpha_i,\beta_i]}(x_i).$$

Our goal is to derive a Newton-type method for $\min_{x\in\mathbb{R}^n} J(x)$ based on solving the equation $\mathcal{F}(x) = 0$ where $\mathcal{F}$ is the projected Gauß-Seidel step in correction form given by

$$\mathcal{F}(x) = (L + D + \partial\varphi)^{-1}(b - Rx) - x.$$

Here L, D and R are the left/diagonal/right matrices from the splitting $A = L + D + R$.

a) Prove that $\mathcal{F}$ is equivalently given by

$$\mathcal{F}(x) = (D + \partial\varphi)^{-1} (b - (R + L)x - L\mathcal{F}(x)) - x. \tag{1}$$

b) Now we define the functions $f_i := (D_{ii} + \partial\varphi_i)^{-1} \colon \mathbb{R} \to \mathbb{R}$ and $f(x) = (f_i(x_i))_{i=1,\dots,n}$. Prove that in the sense of Clarke

$$\partial f_i(x_i) = \begin{cases} \left[0, D_{ii}^{-1}\right] & \text{if } f_i(x_i) \in \{\alpha_i, \beta_i\}, \\ \{D_{ii}^{-1}\} & \text{else.} \end{cases} \tag{2}$$

We define $r := b - (R + L)x - L\mathcal{F}(x)$ and write $\mathcal{F}$ using (1) as

$$\mathcal{F}(x) = f(r)\,(b - (R + L)x - L\mathcal{F}(x)) - x.$$

Assuming that a chain rule holds for the generalized differential ∂ we obtain

$$\partial\mathcal{F}(x) = \partial f(r)\,(-(R + L) - L\partial\mathcal{F}(x)) - I.$$

For some fixed $x \in \mathbb{R}^n$ we define $\mathcal{I} := \{i \in \{1,\dots,n\} \mid (x + \mathcal{F}(x))_i \in (\alpha_i, \beta_i)\}$.

Please turn over...

c) From now on identify $\partial f(x)_i$ with 0 if $0 \in \partial f(x)_i$ and else identify $\partial f(x)_i$ with D_{ii}^{-1}. Show that

$$\partial \mathcal{F}(x) = -D_{\mathcal{I}}^{+} \left(R + L + L\partial \mathcal{F}(x)\right) - I = -(D + L)_{\mathcal{I}}^{+} R_{\mathcal{I}} - I \tag{3}$$

where $(\cdot)^{+}$ denotes the Moore-Penrose pseudoinverse.

d) Prove

$$\nabla J \left(x + \mathcal{F}(x)\right)_{\mathcal{I}} = R_{\mathcal{I}} \mathcal{F}(x). \tag{4}$$

e) Show that

$$y = x + \mathcal{F}(x) - A_{\mathcal{I}}^{+} \nabla J \left(x + \mathcal{F}(x)\right) \tag{5}$$

is equivalent to $y = x + \Delta y$ with Δy given by

$$\partial \mathcal{F}(x) \Delta y = -\mathcal{F}(x). \tag{6}$$

Have fun!