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Numerics II
WS 2017/18

Name: ______________________ Matr.-Nr.: ______________

Course of studies: ◯ Mathematics ◯ Bioinformatics ◯ BMS ◯ Computer Science
◯ other:

Intended degree: ◯ Diplom ◯ Lehramt (Staatsexamen) ◯ other:
◯ Bachelor (Mono) ◯ Bachelor (Kombi, Lehramt) ◯ Master

Note your name on all sheets you hand in and staple them together. Please do not use a pencil. You are allowed to use all your written documents, books, and a non-programmable calculator. Other electronic devices are not allowed. The exam consists of 3 pages.

If you want to find your results on the lecture web page next to your matriculation number sign the following declaration:

I agree with the publication of my results next to my matriculation number on the lecture web page.

______________________ (sign here)

Problem	1	2	3	4	$\sum$
Points					

Good luck!

Complete **all** of the following exercises!

Problem 1 (8 points)
For each of the following statements check if it is 'true' or 'false', note your answer, and explain it by one sentence or a counterexample.

You get one point for each statement where your answer and explanation is correct. If an answer is not correct or no correct explanation is given, you get zero points for the corresponding statement.

a) Consider a linear system $Ax^* = b$ with symmetric, positive definite $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $J(y) = \frac{1}{2}\langle Ay, y\rangle - \langle b, y\rangle$. For any $x^0 \in \mathbb{R}^n$, the iterates $x^0, x^1, x^2, \dots$ provided by the symmetric Gauß–Seidel method satisfy $J(x^0) \geq J(x^1) \geq J(x^2) \geq \cdots \geq J(x^*)$.

b) The discrete flow operator Ψ^τ of a Gaußmethod for an ODE with autonomous, dissipative right hand side is Lipschitz-continuous for all $\tau > 0$.

c) All Gaußmethods are B-stable.

d) All Runge–Kutta methods are A-stable.

e) For any solution $x : \mathbb{R} \to \mathbb{R}$ of the ODE

$$x'' = -x$$

the function $t \mapsto x(t)^2 + (x'(t))^2$ is constant.

f) The linear iterative method $x^{k+1} = x^k + (b - Ax^k)$ for a symmetric, positive definite matrix $A \in \mathbb{R}^{n\times n}$ converges to the solution x^* of $Ax^* = b$ if

$$\langle Ax, x\rangle \leq \|x\|^2 \qquad \forall x \in \mathbb{R}^n.$$

g) The initial value problem

$$x'(t) = \frac{1}{1 + (x(t))^2} \qquad \forall t \in (0, T], \qquad x(0) = x_0$$

has a unique solution for all $T > 0$ and all initial values $x_0 \in \mathbb{R}$.

h) All Runge–Kutta methods are symplectic.

Please turn over!

Problem 2 (2+2+2 points)

a) Compute all stable and asymptotically stable fixed points $x^* \in \mathbb{R}^2$ of the ODE

$$x' = f(x), \qquad \text{where } f(x) = \begin{pmatrix} -x_1(x_2^2+1) \\ x_1x_2 + 1 - x_2^3 \end{pmatrix}.$$

b) Compute the stability function of Runge's method, i.e., the Runge–Kutta method given by the Butcher scheme

$$\begin{array}{c|cc} & 0 & 0 \\ & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}.$$

c) Which time step restriction guarantees that Runge's method inherits the stability properties of the following linear ODE?

$$x' = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} x$$

Problem 3 (1+2+1 points)

Consider the linear system $Ax^* = b$ for $b \in \mathbb{R}^2$ and the symmetric matrix

$$A = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Assume that $a, c, d \in \mathbb{R}$ are selected such that the matrix is positive definite.

a) Show that the matrix

$$B = \begin{pmatrix} a+1 & 0 \\ 0 & d+c^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

is symmetric, positive definite.

b) Show that

$$\langle Ax, x\rangle \le \langle Bx, x\rangle \quad \forall x \in \mathbb{R}^2,$$

where A and B are given above.

c) Show that the linear iteration $x^{k+1} = x^k + \omega B^{-1}(b - Ax^k)$ converges to the solution x^* for any $\omega \in (0,2)$ and B as given above.

Problem 4 (2+2+2 points)

Consider the ODE

$$x'' = \sin(x) + \cos(x) \tag{1}$$

for a sufficiently smooth function $x : \mathbb{R} \to \mathbb{R}$.

a) Rewrite (1) as a Hamiltonian system with Hamiltonian $H(p,q)$.

b) Show that $H(p,q)$ is conserved throughout the evolution, i.e., $H(x'(t), x(t))$ is constant in t for any solution x of (1).

c) Show that $x'(t)$ is bounded independently of t for any solution x of (1).

End of the exam