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Exercise 4 for the lecture

Numerics II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Tue, 14-11-2017

Problem 1 (4 TP)
Let $B \in \mathbb{R}^{d\times d}$ and $\rho(B) = \max_{\lambda\in\sigma(B)} |\lambda|$ the spectral radius of B. Prove the following characterization of (asymptotic) stability of the linear recursion $x_{k+1} = Bx_k$.

a) If $\rho(B) \le 1$ and every eigenvalue $\lambda \in \sigma(B)$ with $|\lambda| = 1$ fulfills $r(\lambda) = s(\lambda)$, i.e., algebraic and geometric multiplicity of λ coincide, then the recursion is stable.

b) If $\rho(B) < 1$, then the recursion is asymptotically stable.

Hint: Show that any Jordan block $J = \lambda I + N$ satisfies $\|J^k\| \le |\lambda|^k p(k)$ for some polynomial p and use this to show the assertions.

Problem 2 (4 TP)
We consider the system $x'(t) = Ax(t)$ with the fixed point $x^* = 0$.

a) Let $x^* = 0$ be asymptotically stable. Then there is a stepsize $\tau > 0$ such that the linear recursion
$$x_{k+1} = (I + \tau A)x_k, \quad k = 0, \dots, \tag{1}$$
is asymptotically stable.

b) Let all eigenvalues of A be complex (not real) and let $x^* = 0$ be stable and not asymptotically stable. Then the linear recursion is unstable for all $\tau > 0$.

c) Illustrate the result of b) in the special case
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
by computing explicit Euler approximations with a corresponding `Matlab` programm for the initial value $x_\varepsilon = (\varepsilon, \varepsilon)^T$ with $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$ and suitable final time T and stepsize $\tau > 0$. What happens, if the implicit Euler method is used?

Problem 3 (4 TP)
Approximating the solution of a linear initial value problem

$$x'(t) = Ax(t),\ x(0) = x_0 \in \mathbb{R}^n \tag{2}$$

with implicit Euler scheme results in a linear iteration of the form

$$x_{k+1} = Bx_k,\ x(0) = x_0, \tag{3}$$

where $x_{k+1} \approx x(t_{k+1})$.

a) Prove that if he continuous initial value problem (2) is asymptotically stable, then the linear iteration (3) is also asymptotically stable.

b) Consider the harmonic oscillator problem with

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

Is this system asymptotically stable or just stable? Investigate asymptotically stability and stability of the iteration obtained by applying the implicit Euler method to this problem.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.