Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser, Ana Djurdjevac

Exercise 6 for the lecture

Numerics II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Wed, 29-11-2017

Problem 1 (4 TP)
Compute the Butcher scheme for the collocation method with the supporting points of the Simpson rule.

Problem 2 (4 TP)
The discrete flux $\Psi^{t+\tau,t}x(t)$ (approximating $x(t+\tau)$) of Runge-Kutta methods for non-autonomous initial value problems

$$x'(t) = f(t, x(t)), \quad t > t_0, \qquad x(0) = x_0$$

with $f : \Omega \to \mathbb{R}^d$ is specified by

$$\Psi^{t+\tau,t}x = x + \tau \sum_{i=1}^{s} b_i k_i, \qquad\qquad k_i = f(t + c_i\tau, x + \tau \sum_{j=1}^{s} a_{ij} k_j).$$

The coefficients are given in the extended Butcher scheme

$$\begin{array}{c|c} c & \mathcal{A} \\ \hline & b^T \end{array}.$$

A Runge-Kutta method is called invariant with respect to autonomization if the discrete flux coincides with the discrete flux $\overline{\Psi}^\tau$ of the autonomized system

$$y'(t) = F(y(t)), \quad t > 0, \qquad y(0) = (x_0, t_0)^T$$

with $F : \Omega \times \mathbb{R}^+ \to \mathbb{R}^{d+1}$, $y = (x, t)^T \mapsto F(y) = (f(t, x), 1)^T$, i.e.

$$\overline{\Psi}^\tau \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \Psi^{t+\tau,t}x \\ t + \tau \end{pmatrix}.$$

Show that collocation methods are invariant with respect to autonomization.

Problem 3 (6 PP)

a) Implement the (possibly implicit) Runge-Kutta method for the linear system

$$x'(t) = Mx(t), \qquad t \in (\texttt{I}(1), \texttt{I}(2)] \qquad x(\texttt{I}(1)) = \texttt{x}_0$$

in `matlab` as `function` [x, t, k] = RungeKuttaLinear(M, x0, I, tau, b, A), where `M`, `x0`, `I`, and `tau` denote the system matrix, the initial value, the time interval and the step size, respectively and the Butcher scheme is given by `b`, `A`. The returned values `x` ($d \times n$ matrix), `t` ($1 \times n$ vector), and `k` should contain the solution at each time step, the time steps, and the intermediate $d \times s$ vectors k_i for all time steps, respectively.

b) Test your program with the initial value problem

$$x'(t) = -\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x(t), \qquad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1}$$

on the interval $(0, 5]$ with the step sizes $\tau = 10^{-3}, 10^{-2}, 10^{-1}, 1$ for the method of Runge, and the Gauß method of order 4. Plot the discretization error and discuss the numerical results.

c) Evaluate the collocation polynomials u of the Gauß method of order 4 applied to the above problem with step size $\tau = 1$ from the intermediate vectors k_i on a sufficiently fine sample grid and plot the discrete trajectories given by the values of u.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.