Fachbereich Mathematik & Informatik
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Exercise 7 for the lecture

NUMERICS II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Tue, 4-12-2017

Problem 1 (4 TP)
Prove the following remarks:

a) Let f be dissipativ. Then every fixed point of $x'(t) = f(x)$ is stable.

b) If f satisfies the stronger condition

$$(f(x) - f(y), x - y) \leq \mu |x - y|^2 \qquad \mu < 0,$$

then every fixed point is assymptotically stable.

Problem 2 (2 TP)
Let $E : \mathbb{R}^n \to \mathbb{R}$ be a continuous and coercive functional.

a) Show that E has a minimizer in $\mathbb{R}^n$.

b) Show that the minimizer is unique if E is additionally strictly convex.

c) Show that E does not necessarily have a minimizer if coercivity is dropped. Recall that the functional E is called coercive if

$$\lim_{\|x\| \to \infty} E(x) = \infty.$$

Problem 3 (8 TP)
Let $E : \mathbb{R}^n \to \mathbb{R}$ be a convex functional and consider the associated *gradient flow*

$$Mx'(t) = -\nabla E(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $\nabla E(x(t)) \in \mathbb{R}^n$ is the gradient of E at $x(t)$ and $M \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

a) Show that $E(x(t)) \leq E(x_0)$ for all $t > 0$. Show then that even $E(x(t)) < E(x_0)$ if $\nabla E(x_0) \neq 0$.

b) Show that $f(x) = -M^{-1}\nabla E(x)$ is dissipative with respect to the M - scalar product $\langle x, y \rangle_M := x^T My$.

c) Show that $x^* \in \mathbb{R}^n$ is a fixed point of (1) iff (= if and only if) x^* is a minimum of E. Furthermore, show that each isolated fixed point of (1) is stable.

d) Assume that E is strictly convex. Show that each fixed point of (1) is asymptotically stable.

Hint: First prove the properties for the case when M is identity matrix. For the general case, use the inner product induced with the matrix M or M^{-1} where convenient.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.