Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser, Ana Djurdjevac

Exercise 8 for the lecture

Numerics II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Wed, 13-12-2017

Problem 1 (4 TP)

A Butcher-scheme $\begin{array}{c|c} & \mathbb{A} \\ \hline & b^T \end{array}$ with

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \dots & a_{ss} \end{pmatrix}$$

defines a *diagonally implicit* Runge-Kutta (DIRK) method Ψ^τ of stage s. If $a_{11} = a_{22} = \ldots = a_{ss}$, Ψ^τ is called a *singly diagonally implicit* Runge-Kutta (SDIRK) method.

a) Describe the advantages of DIRK and SDIRK methods, over usual implicit Runge-Kutta methods.

b) Construct a 2-stage SDIRK method $\Psi^\tau_{2,3}$ of order $p = 3$.

c) Is $\Psi^\tau_{2,3}$ A-stable?

Problem 2 (4 TP)

Consider the *heat equation*

$$\frac{d}{dt}u(x,t) = \Delta u(x,t) \tag{1}$$

with $u : [a,b] \times \mathbb{R}_0^+ \to \mathbb{R}$, the boundary conditions $u(a,t) = u(b,t) = 0$ and the initial condition $u(x,0) = u_0(x)$. Let there be an equidistant partition $a < x_1 < \ldots < x_n < b$ of the interval $[a,b]$, i.e.,

$$x_i = a + \frac{i(b-a)}{n+1}, \qquad i = 1, \ldots, n.$$

The quantity $h = (b-a)/(n+1)$ is called the *grid size*.

a) Discretize (1) by central difference quotients at the points x_i. Write the spatially discrete problem as

$$u_h'(t) = -A_h u_h(t), \qquad u_h(0) = u_{h,0}$$

with $u_h(t) \in \mathbb{R}^n$ and give $u_{h,0}$ and the matrix A_h.

b) Show that there is a functional $E_h : \mathbb{R}^n \to \mathbb{R}$ such that

$$u_h'(t) = -\nabla E_h(u_h(t)).$$

c) Show that E_h is strictly convex.

Problem 3 (4 TP)
Let H be a Hilbert space and $f\colon H \to \mathbb{R}$ Gâteaux-differentiable. The function f is called strongly convex with modulus $\mu > 0$ if for all $x, y \in H$ and $\lambda \in (0,1)$

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)\frac{\mu}{2}\|x-y\|^2.$$

a) Show that f is convex if and only if

$$f(x) - f(y) \ge \langle Df(y), x-y\rangle \qquad \forall x, y \in H. \tag{2}$$

b) Show that f is convex if and only if $Df\colon H \to H'$ is monotone, i. e.,

$$\langle Df(x) - Df(y), x-y\rangle \ge 0 \qquad \forall x, y \in H. \tag{3}$$

c) Show that f is strongly convex with modulus $\mu > 0$ if and only if

$$f(x) - f(y) \ge \langle Df(y), x-y\rangle + \frac{\mu}{2}\|x-y\|^2 \qquad \forall x, y \in H. \tag{4}$$

d) Show that f is strongly convex with modulus $\mu > 0$ if and only if $Df\colon H \to H'$ is strongly monotone, i. e.,

$$\langle Df(x) - Df(y), x-y\rangle \ge \mu\|x-y\|^2 \qquad \forall x, y \in H. \tag{5}$$

Problem 4 (4 TP)
Consider the initial value problem

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \qquad \begin{pmatrix} q \\ p \end{pmatrix}(0) = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{6}$$

Let the flow $\Phi^t x_0$ be the solution of (6) with initial condition $x_0 = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$. Show that the flow map Φ^t conserves the Hamiltonian function

$$H(q,p) = \frac{1}{2}(q^2 + p^2)$$

i.e. for all times t we have

$$H(\Phi^t x_0) = H(x_0).$$

Furthermore, show that neither the explicit nor implicit Euler methods Ψ^τ conserve H i.e. $H(\Psi^\tau x) \neq H(x)$, for any $x \in \mathbb{R}^2$.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.