Fachbereich Mathematik & Informatik
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Exercise 9 for the lecture

Numerics II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Tue, 19-12-2017

Problem 1 (4 TP)

Prove that the so-called symplectic Euler method

$$y_{n+1} = \Psi^\tau y_n = \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = y_n + \tau \begin{pmatrix} -H_q(p_{n+1}, q_n) \\ H_p(p_{n+1}, q_n) \end{pmatrix}$$

is symplectic.

Hint: Consider the numerical flow $\Phi_\tau(p_n, q_n) = (p_{n+1}, q_{n+1})$ and show that it satisfies

$$\begin{pmatrix} I + \tau H_{qp} & 0 \\ -\tau H_{pp} & I \end{pmatrix} \Phi'(p_n, q_n) = \begin{pmatrix} I & -\tau H_{qq} \\ 0 & I + \tau H_{pq} \end{pmatrix},$$

where

$$\Phi'(p_n, q_n) = \begin{pmatrix} \partial_{p_n} p_{n+1} & \partial_{q_n} p_{n+1} \\ \partial_{p_n} q_{n+1} & \partial_{q_n} q_{n+1} \end{pmatrix}.$$

Problem 2 (6 TP)

a) Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^n$ be continuously differentiable such that its Jacobian $Df(y)$ is symmetric for all $y \in D$. Then for every $y_0 \in D$ there exists a neighbourhood of y_0 and a function $H(y)$ such that

$$f(y) = \nabla H(y)$$

on this neighbourhood.

b) Let $f : U \to \mathbb{R}^{2d}$ be a continuously differentiable function, where $U \subset \mathbb{R}^{2d}$ is an open set. Then $y' = f(y)$ is locally Hamiltonian if and only if its flow $\Phi_t(y)$ is symplectic for all $y \in U$ and sufficiently small t.

Problem 3 (4 TP)
Let $\varphi : U \to V$ be a change of coordinates such that φ and φ^{-1} are continuously differentiable functions. If φ is a symplectic transformation, then it preserves the Hamiltonian character of an ODE in the following sense:
The Hamiltonian system $y' = -J\nabla H(y)$ in the new coordinates $z = \Psi(y)$ becomes

$$z' = -J\nabla K(z) \quad \text{where } K(z) = H(\varphi^{-1}(z)). \tag{1}$$

Conversely, if φ transforms every Hamiltonian system to another Hamiltonian system via (1), then φ is symplectic.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.