Fachbereich Mathematik & Informatik
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Exercise 10 for the lecture

Numerics II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Tue, 09-01-2018

Problem 1 (5 TP)

a) Assume that $A \in \mathbb{R}^{n\times n}$ is strongly diagonal dominant, i.e.

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \qquad \forall i = 1, \dots, n. \tag{1}$$

Prove that the Jacobi iteration is globally convergent.

b) Is (1) a necessary condition for global convergence?

c) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, with $\|A\| < 1$. Show that for spectral radius of A it holds $\rho(A) < 1$.

d) Let $A \in \mathbb{R}^{n\times n}$ be symmetric with $\rho(A) < 1$. Show that for some norm on $\mathbb{R}^n$ it holds that $\|A\| < 1$.

Problem 2 (5+2 TP)

a) Let $M \in \mathbb{R}^{n\times n}$ and let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product in $\mathbb{R}^n$. Define

$$\langle x, y \rangle_M := < Mx, y >, \quad x, y \in \mathbb{R}^n.$$

Find necessary and sufficient conditions on M such that $\langle \cdot, \cdot \rangle_M$ is a scalar product.

b) Let $B \in \mathbb{R}^{n\times n}$ be a preconditioner satisfying

$$\mu_0 \langle Bx, x \rangle \leq \langle Ax, x \rangle \leq \mu_1 \langle Bx, x \rangle, \quad \forall x \in \mathbb{R}^n$$

for some $0 \leq \mu_0, \mu_1 \in \mathbb{R}$. Show that

$$\kappa(B^{-1/2} A B^{-1/2}) \leq \frac{\mu_1}{\mu_0}$$

holds.

c) (extra points) Find examples of matrices A and B such that

$$\lambda_{\max}(AB)/\lambda_{\min}(AB) = \kappa(AB)$$

does not hold.

Problem 3 (3 PP)
Consider the following linear system

$$\begin{pmatrix} 3 & -1 & & 0 \\ -2 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -2 & 3 \end{pmatrix} x = b,$$

where A is 10×10 tridiagonal matrix and b is chosen in a such a way that the solution is a unit vector $(1, \ldots, 1)^T$. Using `Matlab` calculate the spectral radius of the iteration matrices of Jacobi and Gauss-Seidel method $Id - B^{-1}A$. Plot the eigenvalues of these two matrices. What can you conclude about the convergence rates of Jacobi and Gauss-Seidel for this example?

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.