Fachbereich Mathematik & Informatik
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Exercise 11 for the lecture

Numerics II

WS 2017/2018

http://numerik.mi.fu-berlin.de/wiki/WS_2017/NumericsII.php

Due: Wed, 17-01-2018

Problem 1 (3 TP)
Let

$$B = (D + L)D^{-1}(D + L)^T$$

be the symmetric Gauß-Seidel preconditioner for the s.p.d. matrix $A = D + L + R$. Prove that the following so-called smoothing property holds

$$\langle Ax, x\rangle \leq \omega_0 \langle Bx, x\rangle \quad \forall x$$

with $\omega_0 = 1$.

Problem 2 (5 TP)
Let $C, A \in \mathbb{R}^{n\times n}$, C is an s.p.d. matrix and A is symmetric with respect to $\langle \cdot, \cdot\rangle_C$.

a) Show that CA is symmetric.

b) Show that $C^{1/2}AC^{-1/2}$ is symmetric.

c) Show that there exist $T, \hat{T}, D \in \mathbb{R}^{n\times n}$, where D is diagonal, T is regular, and $\hat{T}$ is orthogonal matrix such that

$$C^{1/2}AC^{-1/2} = \hat{T}D\hat{T}^T$$

and

$$A = TDT^{-1}.$$

d) Show that

$$\lambda_{\min}(A) = \min_{x\neq 0} \frac{\langle Ax, x\rangle_C}{\langle x, x\rangle_C} \leq \max_{x\neq 0} \frac{\langle Ax, x\rangle_C}{\langle x, x\rangle_C} = \lambda_{\max}(A).$$

Problem 3 *[extra points]* (4 TP)
Prove the following remarks:

a) If matrix $A \in \mathbb{R}^{n\times n}$ is strongly diagonal dominant, i.e.,

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \qquad \forall i = 1, \dots, n,$$

then A is regular.

b) Let $A = (a_{ij})_{i,j} \in \mathbb{R}^{n\times n}$. Then for every eigenvalue $\lambda \in \sigma(A)$ there exists some index i such that

$$|\lambda - a_{ii}| \leq r_i := \sum_{j=1, j\neq i}^{n} |a_{ij}|,$$

or, equivalently,

$$\lambda \in \overline{B_{r_i}(a_{ii})} = \{x \in \mathbb{R}^n \mid |x - a_{ii}| \leq r_i\}$$

holds. (This result is also called Gershgorin's theorem and the $\overline{B_{r_i}(a_{ii})}$ are called Gershgorin-circles.)

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.