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Exercise 1 for the lecture

Numerics II

WS 2019/2020

http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php

Due: Thursday, October 24th at the tutorial

1. Aufgabe (3 TP)

For a given matrix $A \in \mathbb{R}^{m\times m}$, consider the following differential equation

$$y'(t) = Ay(t). \tag{1}$$

a) We will denote by $x\colon [0,T] \to \mathbb{R}^m$ and $y\colon [0,T] \to \mathbb{R}^m$ the solutions of (1). For given $a, b \in \mathbb{R}$, prove that $z = ax + by$ is also a solution of (1).

b) For a given $c \in \mathbb{R}_{>0}$ we will denote by A the matrix

$$A = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}.$$

Prove that

$$x(t) = \begin{pmatrix} \sin(ct) \\ \cos(ct) \end{pmatrix}, \qquad y(t) = \begin{pmatrix} -\cos(ct) \\ \sin(ct) \end{pmatrix}$$

are solutions of (1).

c) Let A be the same matrix as in b). Further let $y_0 \in \mathbb{R}^2$ be given. Find a function y, which solves (1) with $y(0) = y_0$.

2. Aufgabe (3TP + 3TP)

Let $f \in C^2(\Omega)$ and for every given initial value $x_0 \in \mathbb{R}^d$, we assume the unique solution $x \in C^2[0,T]$ exists for the initial value problem

$$x'(t) = f(x(t)), \quad t \in (0,T], \quad x(0) = x_0,$$

with $f : \mathbb{R}^d \to \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$.

a) Prove or disprove that the implicit Euler method

$$x_{k+1} = \psi_{im}^{\tau_k} x_k, \quad k = 0, 1, \dots$$

with a given initial iterate $x_0 \in \mathbb{R}^d$ is consistent with order $p = 1$, where we denote by

$$\psi_{im}^{\tau} x := x + \tau f(\psi_{im}^{\tau} x), \quad \tau < \tau_0,$$

the discrete flow operator w.r.t. the time step τ.

b) The implicit mid-point rule is given by

$$x_{k+1} = \psi_{mid}^{\tau_k} x_k, \quad k = 0, 1, \dots$$

with a given initial iterate $x_0 \in \mathbb{R}^d$, where we denote by

$$\psi_{mid}^{\tau} x := x + \tau \frac{1}{2}(k_1 + k_2), \quad k_1 = f(x), \quad k_2 = f(x + \tau \frac{1}{2}(k_1 + k_2)).$$

the discrete flow operator w.r.t. the time step τ.

Give the order of consistency and the Butcher tableau of this method.

3. Aufgabe (4TP)

We consider the initial value problem

$$x' = -\lambda x, \qquad x(0) = x_0$$

with $\lambda > 0$ and the solution $x(t) = x_0 e^{-\lambda t}$

a) Show that the associated flow operator ϕ^t has the property

$$|\phi^t x - \phi^t y| \leq |x - y| \qquad \forall x, y \in \mathbb{R}.$$

b) Let ψ_{ex}^{τ} and ψ_{im}^{τ} be the discrete flows of the explicit and implicit Euler method as applied to the above ode, respectively. Derive sufficient conditions for the stability estimate

$$|\psi^{\tau} x - \psi^{\tau} y| \leq |x - y| \qquad \forall x, y \in \mathbb{R}$$

for the case $\psi^{\tau} = \psi_{\text{ex}}^{\tau}$ and for the case $\psi^{\tau} = \psi_{\text{im}}^{\tau}$.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.