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Exercise 2 for the lecture

Numerics II

WS 2019/2020

http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php

Due: Thursday, October 31th at the tutorial

1. Exercise (4 TP)
Let $A \in \mathbb{R}^{d\times d}$ and $x^* = 0$ is a fixed point. Prove the following statements.

a) If $\nu(A) > 0$, then x^* is not stable.

b) If $\nu(A) \leq 0$ and there exists a $\lambda \in \sigma(A)$ such that $\mathcal{R}(\lambda) = 0$ but $r(\lambda) < s(\lambda)$, then x^* is not stable.

c) If $\nu(A) \leq 0$ and there exists a $\lambda \in \sigma(A)$ such that $\mathcal{R}(\lambda) = 0$ and $r(\lambda) = s(\lambda)$, then x^* is not asymptotically stable.

2. Exercise (6 TP)
We consider the initial value problem

$$x'(t) = -\lambda x(t),\ t > 0, \qquad x(0) = x_0 > 0$$

with $\lambda > 0$.
a) Show that

$$0 \leq x(t) \leq x_0 \qquad \forall t \geq 0.$$

b) Replacing the derivative by the central difference quotient of second order, we obtain the two-step method

$$x_{k+1} = x_{k-1} - 2\tau\lambda x_k, \qquad k = 1, \dots$$

with given x_0 and x_1. Show that

$$x_k = \alpha\mu_1^k + \beta\mu_2^k, \qquad k = 0, \dots$$

holds with $\mu_1 = -\tau\lambda + \sqrt{\tau^2\lambda^2 + 1}$, $\mu_2 = -\tau\lambda - \sqrt{\tau^2\lambda^2 + 1}$ and α, β solving the linear system

$$\alpha + \beta = x_0, \quad \alpha\mu_1 + \beta\mu_2 = x_1.$$

c) Now fix x_0 and $x_1 = x_0 - \tau\lambda x_0$. Is it possible to find problem parameters $\lambda > 0$, $x_0 > 0$, and discretization parameter $\tau > 0$ that provide the same qualitative behavior as the exact solution, i.e.,

$$0 \leq x_k \leq x_0 \qquad \forall k = 0, 1, \ldots?$$

What qualitative behavior of the approximate solution is occurring for $x_0 = 1$?
d) Numerically illustrate the theoretical results for $\lambda = 1$ the intervall $[0, 1]$, step size $\tau = 1/n$, and $n = 10, 100, 1000$ by plotting the correponding approximations x_k over $t_k = k\tau$ for $k = 1, \ldots, n$. What do you observe for decreasing τ?

3. Exercise (4 TP)
Calculate the explicit solutions of the following linear differential equations

a)

$$\dot{x} = \begin{pmatrix} -3 & 0 & 2 \\ -1 & -3 & 5 \\ -1 & 0 & 0 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{1}$$

b)

$$\dot{x} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{2}$$

Hint: Find appropriate transformations T, s.t. the system matrix is transformed into a block diagonal form.

4. Exercise (2 TP)
Let Φ^t be a flow on $\mathbb{R}^d$. Let x_0 be stable in the following sense:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in X \;\; (|x - x_0| < \delta \;\; \Rightarrow \;\; \forall t > 0 \;\; |\Phi^t(x) - \Phi^t(x_0)| < \varepsilon).$$

Prove or disprove: x_0 is a fixpoint of the flow.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.