Fachbereich Mathematik & Informatik
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Exercise 6 for the lecture

Numerics II

WS 2019/2020

http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php

Due: Thursday, November 28th at the tutorial

1. Exercise (2TP + 2TP)
Consider

$$x' = f(x), \tag{1}$$

with $f : \mathbb{R}^d \to \mathbb{R}^d$ and prove the following statements.

a) If f is dissipative, then every fixed point of (1) is stable.

b) If f is strictly dissipative w.r.t. the scalar product $\langle \cdot, \cdot \rangle$ in the sense that there exists a constant $\mu > 0$, such that

$$\langle f(x) - f(\bar{x}), x - \bar{x} \rangle \leq -\mu |x - \bar{x}|^2 \qquad \forall x, \bar{x} \in \mathbb{R}^d \tag{2}$$

holds, then every fixed point of f is asymptotically stable. Hint: Use Gårdings inequality to show that

$$|\phi^t x - \phi^t \bar{x}| \leq e^{-\mu t} |x - \bar{x}| \tag{3}$$

holds for $t \geq 0$ and all $x, \bar{x} \in \mathbb{R}^d$.

2. Exercise (4TP)
Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable and have the fixed point x^*. Show that

$$\rho(D\psi(x^*)) < 1$$

implies that x^* is an asymptotically stable fixed point of the recursion $x_{k+1} = \psi(x_k)$, $k = 0, 1, \dots$.

3. Exercise (4TP)
Let $\Psi^\tau : \mathbb{R}^n \to \mathbb{R}^n$ the discrete flow operator of the implicit trapezoidal rule with stepsize τ as applied to the linear system

$$x'(t) = Ax(t).$$

a) Show that Ψ^τ can be written as

$$\Psi^\tau = R(\tau A),$$

with a rational function R of the matrix τA.

b) Derive sufficient conditions on τ for the A-stability of Ψ^τ. Is asymptotic stability inherited from the continuous problem?

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.