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Exercise 7 for the lecture

# Numerics II

WS 2019/2020

http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php

**Due: Thursday, December 05th at the tutorial**

**1. Exercise** (4TP)

The discrete flow $\Psi^{t+\tau,t}x(t)$ (approximating $x(t+\tau)$) of Runge-Kutta methods for non-autonomous initial value problems

$$x'(t) = f(t, x(t)), \quad t > t_0, \qquad x(0) = x_0$$

with $f : \Omega \to \mathbb{R}^d$ is specified by

$$\Psi^{t+\tau,t}x = x + \tau \sum_{i=1}^{s} b_i k_i, \qquad k_i = f(t + c_i\tau, x + \tau \sum_{j=1}^{s} a_{ij} k_j).$$

The coefficients are given in the extended Butcher scheme

$$\begin{array}{c|c} c & \mathcal{A} \\ \hline & b^T \end{array}.$$

A Runge-Kutta method is called invariant with respect to autonomization if the discrete flow coincides with the discrete flow $\overline{\Psi}^\tau$ of the autonomized system

$$y'(t) = F(y(t)), \quad t > 0, \qquad y(0) = (x_0, t_0)^T$$

with $F : \Omega \times \mathbb{R}^+ \to \mathbb{R}^{d+1}$, $y = (x, t)^T \mapsto F(y) = (f(t, x), 1)^T$, i.e.

$$\overline{\Psi}^\tau \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \Psi^{t+\tau,t}x \\ t + \tau \end{pmatrix}.$$

Show that collocation methods are invariant with respect to autonomization.

**2. Exercise** (5TP)

A Butcher-scheme $\begin{array}{c|c} & \mathbb{A} \\ \hline & b^T \end{array}$ with

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \dots & a_{ss} \end{pmatrix}$$

defines a *diagonally implicit* Runge-Kutta (DIRK) method $\Psi^\tau$ of stage $s$. If $a_{11} = a_{22} = \ldots = a_{ss}$, $\Psi^\tau$ is called a *singly diagonally implicit* Runge-Kutta (SDIRK) method.

a) Describe the advantages of DIRK and SDIRK methods, over usual implicit Runge-Kutta methods.

b) Construct a 2-stage SDIRK method $\Psi^\tau_{2,3}$ of order $p = 3$.

c) Is $\Psi^\tau_{2,3}$ $A$-stable?

**3. Exercise** (4TP)
Consider the scalar differential equation

$$x' = \lambda(1 - x^2), \qquad \lambda > 0. \tag{1}$$

a) Show that $x^*_s = 1$ is an asymptotically stable and $x^*_u = -1$ an unstable fixed point of (1).

b) Compute the time step restriction for the explicit euler method applied to the linearized problem

$$x' = -2\lambda(x - 1),$$

such that $x^*_s$ is an asymptotically stable fixed point of the resulting discrete linear problem.

c) Is the time step restriction from b) also sufficient to guarantee that $x^*_s$ is an asymptotically stable fixed point of the discrete nonlinear problem, resulting from aplly-ing the explicit euler method to (1)?

## General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.