Fachbereich Mathematik & Informatik
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Exercise 8 for the lecture

Numerics II

WS 2019/2020

http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php

Due: Thursday, December 12th at the tutorial

1. Exercise (5TP)
Let $E : \mathbb{R}^n \to \mathbb{R}$ be a convex functional and consider the associated *gradient flow*

$$x'(t) = -\nabla E(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $\nabla E(x(t)) \in \mathbb{R}^n$ is the gradient of E at $x(t)$.

a) Show that $E(x(t)) \leq E(x_0)$ for all $t > 0$. Show then that even $E(x(t)) < E(x_0)$ if $\nabla E(x_0) \neq 0$.

b) Show that $f(x) = -\nabla E(x)$ is dissipative with respect to the Euclidean scalar product in $\mathbb{R}^n$.

c) Show that $x^* \in \mathbb{R}^n$ is a fixed point of (1) iff (= if and only if) x^* is a minimum of E. Furthermore, show that each isolated fixed point of (1) is stable.

d) Assume that E is strictly convex. Show that each fixed point of (1) is asymptotically stable.

2. Exercise (3TP)
Consider the energy functional

$$E(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - bv)dx, \qquad v \in C_0^1(\overline{\Omega}), \quad b \in C(\overline{\Omega}).$$

Show that the gradient of E at $u \in C_0^1(\overline{\Omega})$ is given by

$$\nabla E(u)(v) = (\nabla u, \nabla v) - (b, v), \qquad v \in C_0^1(\overline{\Omega}),$$

where $(\cdot,\cdot)$ denotes the L^2 scalar product.

3. Exercise (6TP)
Consider the *heat equation*

$$\frac{d}{dt}u(x,t) = \Delta u(x,t) \tag{2}$$

with $u : [a, b] \times \mathbb{R}_0^+ \to \mathbb{R}$, the boundary conditions $u(a, t) = u(b, t) = 0$ and the initial condition $u(x, 0) = u_0(x)$. Let there be an equidistant partition $a < x_1 < \ldots < x_n < b$ of the interval $[a, b]$, i.e.,

$$x_i = a + \frac{i(b-a)}{n+1}, \qquad i = 1, \ldots, n.$$

The quantity $h = (b-a)/(n+1)$ is called the *grid size.*

a) Discretize (2) by central difference quotients at the points x_i. Write the spatially discrete problem as

$$u_h'(t) = -A_h u_h(t), \qquad u_h(0) = u_{h,0}$$

with $u_h(t) \in \mathbb{R}^n$ and give $u_{h,0}$ and the matrix A_h.

b) Show that there is a functional $E_h : \mathbb{R}^n \to \mathbb{R}$ such that

$$u_h'(t) = -\nabla E_h(u_h(t)).$$

c) Show that E_h is strictly convex.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.