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Exercise 9 for the lecture

Numerics II

WS 2019/2020

http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php

Due: Thursday, December 19th at the tutorial

1. Exercise (2TP + 2TP)
Consider the differential algebraic system

$$\begin{pmatrix} 0 & 0 & 0 \\ c & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} x' = \begin{pmatrix} 1 & 0 & -1 \\ c & -\frac{1}{R} & \frac{1}{R} \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} -u \\ 0 \\ 0 \end{pmatrix}, \qquad x(0) = x_0 \tag{1}$$

with $c, R, u > 0$.

a) Rewrite this system in normal form, i.e., as

$$\begin{aligned} y'(t) &= Jy(t) + f, & y(0) &= y_0, \\ Nz'(t) &= z(t) + g, & z(0) &= z_0. \end{aligned}$$

Give f, g, J, N, y_0, z_0, and the transformation $T : x \mapsto (y, z)$.

b) How must x_0 be chosen such that there is a unique solution of (1) ?

2. Exercise (2TP)
Let $E : \mathbb{R} \to \mathbb{R}$ be strictly convex. Show that coercivity of E then is necessary and sufficient for the existence of a minimizer of E on R.

3. Exercise (3TP)
The pairs (E_1, A_1) and (E_2, A_2) are equivalent, i.e. $(E_1, A_1) \sim (E_2, A_2)$, if there exist $P, Q \in \mathbb{R}^{d \times d}$ regular such that $E_2 = PE_1Q$ a nd $A_2 = PA_1Q$.
Show that the relation $\sim$ is an equivalence relation.

4. Exercise (4PP+2PP+2PP+1PP)
Consider the following nonlinear initial value problem

$$x'(t) = f(x), \qquad t > 0, \qquad x(0) = x_0,$$

with $x(t) \in \mathbb{R}^3$ and

$$f(x) = \begin{pmatrix} -c_1 x_1 + c_3 x_2 x_3 \\ c_1 x_1 - c_2 x_2^2 - c_3 x_2 x_3 \\ c_2 x_2^2 \end{pmatrix}.$$

a) Implement an implicit Runge-Kutta method for this equation in MATLAB as `function [x, t] = RungeKuttaNewton(f, Df, x0, I, tau, b, A, TOL)`, where `f`, `Df`, `x0`, `I`, and `tau` denote the function implementing the right hand side f, the Jacobian Df, the initial value, the time interval, the step size, and the tolerance for the nonlinear solver respectively and the Butcher scheme is given by `b, A`. The returned values `x`, `t` should contain the solution at each time step and the time steps, respectively.

   Use the fixed point iteration, Newton iteration and the simplified Newton method and the stopping criterion presented in the lecture to solve the nonlinear system

$$F(Z) = Z - \tau \begin{pmatrix} \sum_{j=1}^{s} a_{1j} f(x + z_j) \\ \vdots \\ \sum_{j=1}^{s} a_{sj} f(x + z_j) \end{pmatrix} = 0,$$

   with $Z = [z_1, \ldots, z_s]^T \in \mathbb{R}^{3s}$.

b) Test your code for the initial value $x_0 = (1, 0, 0)^T$ and $x_0 = (0, 0, 0)^T$, the time interval $[0, 10]$, and the parameter vector $c = (1, 2, 3)$. Use the implicit Euler method, the implicit midpoint rule and the Gauß-method of order 6 for your tests. Estimate the discretization error by computing an sufficiently good reference solution numerically. Plot the estimated error as a function of the step size for all methods and for an appropiate set of step sizes. What convergence orders do you observe?

c) Investigate the efficiency of the given non linear solvers by counting the numbers of function evaluations. Use tests from previous task.

d) Plot the total mass $x_1 + x_2 + x_3$ as function of the time. Show that the continuous flux and every Runge-Kutta scheme preserve the total mass.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.