Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Xingjian Zhang-Schneider

Exercise 12 for the lecture

Numerics II

WS 2019/2020

`http://numerik.mi.fu-berlin.de/wiki/WS_2019/NumericsII.php`

Due: Thursday, January 23th at the tutorial

1. Exercise (5PP)
Consider the pendulum equation in polar coordinates

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \frac{1}{m} p \\ -m \frac{g}{r_0} \cos q \end{pmatrix} \qquad \begin{pmatrix} q \\ p \end{pmatrix}(0) = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$$

where q, g, m, and r_0 denote the angle, the gravity, the mass and the radius, respectively.

a) Implement the Störmer-Verlet method for this equation in `matlab` as `function [p, q, t] = StörmerVerlet(m, g, r0, p0, q0, I, tau)`, where `(m, g, r0)`, `(p0, q0)`, `I`, and `tau` denote the problem parameters, the initial values, the time interval and the step size, respectively.

b) Test your program with the radius $r_0 = 10cm$, the mass $m = 100g$, and the gravity of the moon. Use the time interval $[0s, 20s]$ with the initial value $p_0 = 0\frac{kgm}{s}$, $q_0 = 0m$ for various time step sizes.

c) Plot the solution in the phase space, the solution in Euclidean coordinates, and the associated Hamiltonian H.

2. Exercise (2TP+3PP)
Consider the the linear system

$$AU = b \tag{1}$$

with the symmetric positive definite matrix $A \in \mathbb{R}^{n,n}$ and $b \in \mathbb{R}^n$.

a) Compute an upper bound for the convergence rate of the Jacobi method applied to the linear system (1) with the matrix A obtained by a finite difference discretization of the Poisson equation using a uniform grid on $[0, 1] \times [0, 1]$ given in the lecture.

b) Implement the Jacobi and the Gauß-Seidel methods in `matlab` as

```
function [u, uk] = Jacobi(A, b, u0, tol, uexact)
```

and

```
function [u, uk] = GaussSeidel(A, b, u0, tol, uexact).
```

`u`, `u`k, `A`, `b`, `u`0, `t`ol, and `u`$exact$ denote the last iterate, a vector containing all iterates, the system matrix, the right hand side, the initial iterate, the error tolerance, and the exact solution, respectively. The iteration should stop if the energy norm $\|\cdot\|_A = \langle A\cdot,\cdot\rangle^{0.5}$ of the error is smaller than the tolerance. Test your programm with the matrix of part a) and the right hand side $b = AU$ where U is the point wise evaluation of $(x_1 - x_1^2)(x_2 - x_2^2)$ for $\mathtt{u0} = 0$, $\mathtt{t}ol = 10^{-8}$ and various choices of n. Plot the error over the number of iteration steps and compute the average convergence rate for each choice of n.

3. Exercise (3TP)
Show that, if A is strongly diagonal dominant, i.e.,

$$\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| < |a_{ii}| \quad \forall i = 1,\dots n,$$

the Jacobi method is globally convergent.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@zedat.fu-berlin.de. with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.