Freie Universität Berlin
Prof. Dr. Ralf Kornhuber

Elliptic partial differential equations with highly oscillating or random coefficients WS 2020/2021

1. Exercise sheet, due December 2, 2020

Problem 1 (5 Points)
Let as consider the following one dimensional problem.

$$-\frac{d}{dx}\left(\alpha\left(\frac{x}{\varepsilon}\right)\frac{du^\varepsilon}{dx}\right) = f \quad \text{in } (0, L)$$
$$u^\varepsilon(0) = u^\varepsilon(L) = 0.$$

We assume that f and α are smooth functions. Furthermore, $\alpha(y)$ is $1-$periodic and there exist constants $0 < \alpha_0 \leq \alpha_1 < \infty$ such that

$$\alpha_0 \leq \alpha(y) \leq \alpha_1, \qquad \text{for every } y \in [0, 1].$$

a) Derive a suitable representation of the homogenized coefficient α^* in terms of α.

b) Prove that the homogenized coefficient α^* satisfies

$$\alpha_0 \leq \alpha^* \leq \alpha_1.$$

Moreover, prove that α^* is bounded from above by the average of $\alpha(y)$:

$$\alpha^* \leq \int_0^1 \alpha(y)dy.$$

Problem 2 (5 Points)
Write a Matlab program for the approximation of the 1-D model problem from the lecture by piecewise linear finite elements. Plot the H^1-error over the uniform mesh size $h = 1, 10^{-1}, \ldots, 10^{-5}$ for $\varepsilon = 10^{-2},\ 10^{-4},\ 10^{-6}$ and comment on the results.

Problem 3 (3 Points)
a) Prove that in a Banach space the strong convergence of a sequence implies weak convergence. Does the inverse implication hold true?

b) Prove that weak convergence of a bounded sequence $\{v_k\}$ in a separable Hilbert space H, is equivalent to convergence of the coefficients of its representation by an orthonormal basis.

c) Let $a_n, b_n \in L^2(\Omega),\ n \in \mathbb{N}$. Prove that $a_n \rightharpoonup a$ and $b_n \to b$ implies $a_n b_n \to ab \quad \text{in } L^1(\Omega)$.