Freie Universität Berlin
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**Elliptic partial differential equations
with highly oscillating or random coefficients
WS 2020/2021**

2. Exercise sheet, due December 16, 2020

Problem 1 (4 Points)
A two-scale approach to the problem

$$\begin{aligned} -\nabla \cdot (\alpha_\varepsilon \nabla u_\varepsilon) &= f, \quad \text{in } \Omega \\ u_\varepsilon &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

was presented in the lecture. In this case we assumed that the coefficient $\alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right)$ depends only on the microscale. What happens if the coefficient depends explicitly on the macroscale and the microscale i.e. $\alpha_\varepsilon(x) = \alpha\left(x, \frac{x}{\varepsilon}\right)$, with $\alpha(x, y)$ being 1−periodic in y and smooth in x.

a) Derive the cell-problem for w in this case.

b) Prove for the 1-dimensional case that the homogenized equation

$$\begin{aligned} -\nabla \cdot (\alpha^* \nabla u) &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

has the homogenized coefficient given by

$$\alpha^*(x) = \int_Y \alpha(x, y)(1 + \nabla_y w(x, y))^2 dy.$$

Problem 2 (4 Points)
a) Show that the cell-problem can be formulated as

$$-\nabla_y \cdot (\alpha \nabla_y (w_l + y_l)) = 0, \quad l = 1, \ldots, d$$

for arbitrary $y \in \mathbb{R}^d$.

b) Show that the following two definitions of the homogenized coefficients are equivalent

i)

$$\alpha^*_{ij} = \int_Y \alpha(y)(\nabla_y w_i(y) + e_i)(\nabla_y w_j(y) + e_j) dy,$$

ii)

$$\alpha^*_{ij} = \int_Y \alpha(y)(\nabla_y w_i(y) + e_i) e_j dy.$$

Problem 3 (4 Points)
Show that the homogenized problem

$$\begin{aligned} -\nabla \cdot (\alpha^* \nabla u) &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

where

$$\alpha^*_{ij} = \int_Y \alpha(y)(\nabla_y w_i(y) + e_i)(\nabla_y w_j(y) + e_j)dy,$$

has a unique weak solution in $H_0^1(\Omega)$, under standard assumptions for the coefficient α (α is $Y-$periodic, uniformly bounded and belongs to $\mathcal{C}^1(\overline{Y})$).