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Elliptic partial differential equations with highly oscillating or random coefficients
WS 2020/2021

4. Exercise sheet, due January 20, 2021

Exercise 1: Clément interpolation (2 + 2 + 2 + 2 + 2 + 2 Points)
Let $\Omega \in \mathbb{R}^d$ be a bounded domain with polygonal boundary. Consider a partition $\mathcal{T}_h$ of Ω into simplices t with maximal diameter h and vertices $\mathcal{N}_h$. The associated space of piecewise linear finite elements $\mathcal{S}_h$ is spanned by the nodal basis λ_p, $p \in \mathcal{N}_h$. For each $p \in \mathcal{N}_h$ let $\omega_p = \operatorname{supp} \lambda_p$ denote the patch consisting of all simplices with common vertex p, the patch ω_t is the union of all simplices that have a common vertex with $t \in \mathcal{T}_h$, and $\mathcal{S}(\omega_p)$ stands for the linear space of restrictions of $v \in \mathcal{S}_h$ to ω_t. For each $p \in \mathcal{N}_h$ and every $w \in L^1(\Omega) \subset L^2(\Omega)$ let

$$w_p \in S(\omega_p): \qquad (w_p, v) = (w, v) \qquad \forall v \in \mathcal{S}(\omega_p)$$

be the L^2-projection on $\mathcal{S}_h(\omega_p)$. Then the Clément interpolation operator $P : L^2(\Omega) \to S_h$ is defined by

$$Pw(x) = \sum_{p \in \mathcal{N}_h} w_p(p)\lambda_p(x), \quad w \in L^2(\Omega).$$

a) Prove the inverse inequality

$$\max_{x \in \omega_p} |v(x)| \lesssim h^{-d/2} \|v\|_{0,\omega_p} \qquad \forall v \in \mathcal{S}(\omega_p), \quad p \in \mathcal{N}_h.$$

Hint: use the transformation to the reference triangle.
b) Prove the local L^∞-stability estimate

$$|w_p(p)| \lesssim h^{-d/2} \|w\|_{0,\omega_p} \qquad \forall w \in L^2(\Omega).$$

c) Prove the inverse estimate

$$\max_{x \in \omega_p} \left| \frac{\partial}{\partial x_k} \lambda_p(x) \right| \lesssim h^{-1} \qquad \forall p \in \mathcal{N}_h,\ k = 1, \dots, d.$$

d) Prove the local L^2-stability and inverse inequality

$$\|Pw\|_{0,t} \lesssim \|w\|_{0,\omega_t} \quad \text{and} \quad |Pw|_{1,t} \lesssim h^{-1} \|w\|_{0,\omega_t} \qquad \forall w \in L^2(\Omega)$$

for all $t \in \mathcal{T}_h$.
e) Prove the local H^1-stability and approximation property

$$|w - Pw|_{1,t} \lesssim |w|_{1,\omega_t} \quad \text{and} \quad \|w - Pw\|_{0,t} \lesssim h|w|_{1,\omega_t} \qquad \forall w \in H^1(\Omega)$$

Hint: Use that for each $w \in H^1(\Omega)$ there is a $w_h \in \mathcal{S}_h(\omega_t)$ such that

$$\|w - w_h\|_{0,\omega_t} \lesssim h|w|_{1,\omega_t} \quad \text{and} \quad |w - w_h|_{1,\omega_t} \lesssim |w|_{1,\omega_t}.$$

This follows from a local Poincaré-type inequality (cf., e.g., Verfürth (1999)).
f) Prove stability and approximation property of the Clément interpolation operator P.