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**Elliptic partial differential equations
with highly oscillating or random coefficients
WS 2020/2021**

5. Exercise sheet, due February 3, 2021

Exercise 1: Multiscale finite elements (2 + 10 Points)
Let $\Omega = (0,1) \in \mathbb{R}^1$ and consider the variational equality

$$u \in H_0^1(\Omega): \quad a(u,v) = \ell(v) \qquad \forall v \in H_0^1(\Omega)$$

where

$$a(v,w) = \int_0^1 \alpha(x) v'(x) w'(x)\, dx, \qquad \ell(v) = \int_0^1 f(x)\, dx.$$

For a given coarse grid $p_i = iH,\ i = 0, \dots N,\ H = 1/N$, and the corresponding partition $\mathcal{T}_H = \{[p_{i-1}, p_i] \mid i = 1, \dots, N\}$ of Ω let

$$\mathcal{S}_H = \{v \in C(\Omega) \cap H_0^1(\Omega) \mid v|_t \text{ affine } \forall t \in \mathcal{T}_H\}$$

denote the space of (classical) piecewise affine finite elements while

$$\mathcal{W} = \text{ span } \{(I - C)\lambda_{p_i} \mid i = 1, \dots, n-1\}$$

stands for the space of modified finite elements induced by the a-orthogonal projection $C : H_0^1(\Omega) \to \ker \Pi$ with $\Pi = I_H$ (nodal interpolation to $\mathcal{S}_H$).
a) Find an explicit basis representation of $\mathcal{V}_h = \mathcal{S}_h \cap \ker \Pi$ with $h = 1/n$ and $n = mN,\ m \geq 1$.
b) Implement a computationally feasible version of the modified finite element method where the evaluation of $C\lambda_{p_i}$ is approximated by

$$\mu_{p_i} \in \mathcal{V}_h \qquad a(\mu_{p_i}, v) = a(\lambda_{p_i}, v) \qquad \forall v \in \mathcal{V}_h$$

and apply the resulting code to the model problem with

$$\alpha(x) = (2 + \sin(2\pi x/\varepsilon))^{-1}, \qquad f \equiv 1,$$

the solution

$$u(x) = x(x-1) + \frac{\varepsilon}{2\pi}\left(x - \frac{1}{2}\right)\left(1 - \cos\left(\frac{2\pi}{\varepsilon}x\right)\right) + \left(\frac{\varepsilon}{2\pi}\right)^2 \left(\sin\left(\frac{2\pi}{\varepsilon}x\right) - \frac{1}{\varepsilon}\right)$$

and the parameters $\varepsilon = 10^{-2}$, $N = 2^j$, $m = 2^{9-j}$ for $j = 1, \dots, 6$ in order to compare the order of convergence with respect to H.