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Elliptic partial differential equations with highly oscillating or random coefficients WS 2020/2021

6. Exercise sheet, due February 17, 2021

Exercise 1.
Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a probability space and let ξ be a random variable. Show that the triple $(I, \mathcal{E}_I, \mathbb{P}_I)$ defined by

$$\begin{aligned} I &= \xi(\Omega) \subset \mathbb{R} \\ \mathcal{E}_I &= B(I) = Bor(\mathbb{R}) \cap I \\ \mathbb{P}_I(A) &= \mathbb{P}(\xi^{-1}(A)) \end{aligned}$$

is a probability space.

Exercise 2.
a) Show that independent random variables ξ and η are uncorrelated.
b) Conversely, are there uncorrelated random variables ξ and η that are not independent?

Exercise 3.
a) Let v be a non-negative random variable. Prove Markov's inequality

$$\mathbb{P}(v \geq \alpha)\alpha \leq \mathbb{E}[v], \qquad \forall \alpha \in \mathbb{R}.$$

b) Prove Chebyshev's inequality

$$\mathbb{P}\left(|v - \mathbb{E}[v]| \geq k\right) \leq \frac{\mathrm{Var}(v)}{k^2}.$$

c) Prove the following 'Law of Large Numbers':

Let $v_i, i = 1, \ldots, n$ be independent identically distributed random variables and $w(\omega_1, \ldots, \omega_n) = \frac{1}{n} \sum_{i=1}^{n} v_i(\omega_i)$. Then $\mathbb{E}[w] = \mathbb{E}[v_i], i = 1, \ldots, n$ and for all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$\mathbb{P}\left(|w - \mathbb{E}[w]| \leq C_\varepsilon n^{-1/2}\right) \geq 1 - \varepsilon.$$