1. Elliptic Multiscale problems

- homogenization
  oscillating periodic coefficients
- multiscale finite elements

2. Elliptic problems with random coefficients

- basics of stochastics
- Monte-Carlo methods
- polynomial chaos
- multilevel Monte-Carlo methods

1. Elliptic multiscale problems

1.1. Weak solutions and finite elements

$$-\operatorname{div}(\alpha \nabla u) = f \quad \text{in } \Omega \in \mathbb{R}^d$$

$$u = 0 \quad \text{on } \partial\Omega$$

weak formulation

(1) $u \in H_0^1(\Omega)$:

$$a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

$$a(u,v) = \int_\Omega \alpha \nabla u \nabla v \, dx$$

$$\ell(v) = \int_\Omega f v \, dx$$

Existence, uniqueness, conditions

assumption:

$$0 < \alpha_0 \leq \alpha(x) \leq \alpha_1 \quad \text{a.e. in } \Omega$$

$$\ell \in H^{-1}(\Omega) = (H_0^1(\Omega))'$$

(sufficient: $f \in L^2(\Omega)$)

consequence:

$a(\cdot,\cdot)$ scalarproduct on $H = H_0^1(\Omega)$

energy norm: $\|v\| = (a(v,v))^{1/2}$

equivalent to $H^1(\Omega)$-norm

$$\|v\|_1 = \left(\|v\|_0^2 + \| |\nabla v| \|_0^2 \right)^{1/2}$$

$$\|v\|_0 = (v,v)^{1/2} = \left(\int_\Omega v^2 \, dx\right)^{1/2}$$

i.e. $\exists \; \gamma, \Gamma > 0$:

$$\gamma \|v\|_1^2 \leq a(v,v), \qquad \forall v, w \in H$$

$$|a(v,w)| \leq \Gamma \|v\|_1 \|w\|_1$$

Theorem 1.1

$\exists$ unique solution $u \in H$ of (1) and we have

$$\frac{\|u - \tilde{u}\|_1}{\|u\|_1} \leq \frac{T}{\gamma} \frac{\|\ell - \tilde{\ell}\|_{-1}}{\|\ell\|_{-1}}$$

$$\tilde{u} \in H : \quad a(\tilde{u}, v) = \tilde{\ell}(v) \quad \forall v \in H$$

$$\|\ell\|_{-1} = \sup_{v \in H} \frac{|\ell(v)|}{\|v\|_1}$$

Proof:

a) existence, uniqueness:

Riesz theorem (Lax-Milgram)

b) condition

$$\|u\|_1^2 \leq \frac{1}{\gamma} \|u\|^2 = \frac{1}{\gamma} a(u,u) \leq |\ell(u)|$$

$$\leq \frac{1}{\gamma} \|\ell\|_{H^{-1}} \cancel{\|u\|_1}$$

$$\|u\|_1 \leq \frac{1}{\gamma} \|\ell\|_{H^{-1}}$$

b) condition

$$\|u\|_1^2 \le \frac{1}{\gamma}\|u\|^2 = \frac{1}{\gamma}a(u,u) \le |\ell(u)|$$

$$\le \frac{1}{\gamma}\|\ell\|_{H^{-1}}\,\|\cancel{u}\|_1$$

$$\|u\|_1 \le \frac{1}{\gamma}\|\ell\|_{H^{-1}}$$

$$a(u-\tilde{u}, v) = (\ell - \tilde{\ell})(v)$$

$$\|u-\tilde{u}\|_1 \le \frac{1}{\gamma}\|\ell-\tilde{\ell}\|_{-1}$$

$$\frac{\|u-\tilde{u}\|_1}{\|u\|_1} \le \frac{1}{\gamma}\frac{\|\ell-\tilde{\ell}\|_{-1}}{\|u\|_1}$$

$$\le \frac{\Gamma}{\gamma}\frac{\|\ell-\tilde{\ell}\|_{-1}}{\|\ell\|_{-1}}$$

by $\quad \|u\|_1 \ge \frac{1}{\Gamma}\|\ell\|_{-1}$

$$|\ell(u)| = |a(u,u)| \le \Gamma\|u\|_1^2$$

$$\frac{|\ell(u)|}{\|u_1\|} \le \Gamma\|u\|_1$$

$$\|\ell\|_{-1} \le \Gamma\|u\|_1$$

Ritz - Galerkin method:

closed subspace $S_h \subset H$

(2) $u_h \in S_h : \quad a(u_h, v) = \ell(v) \quad \forall v \in S_h$

<u>Theorem 1.2</u>

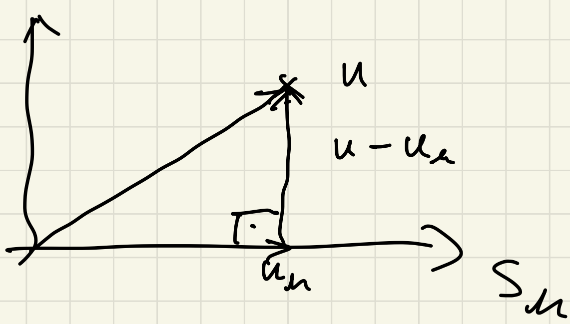
$\exists$ unique solution u_h of (2) and we have

$$\| u - u_h \| \leq \inf_{v \in S_h} \| u - v \|$$

Proof:

a) Galerkin orthogonality

$$a(u - u_h, v) = \ell(v) - \ell(v) = 0 \quad \forall v \in S_h$$



b) $\| u - u_h \|^2 = a(u - u_h, u - u_h)$

$\overset{!}{=} a(u - u_h, u) \overset{!}{=} a(u - u_h, u - v)$

$\leq \| u \cancel{- u_h} \| \, \| u - v \| \quad \forall v \in S_h$

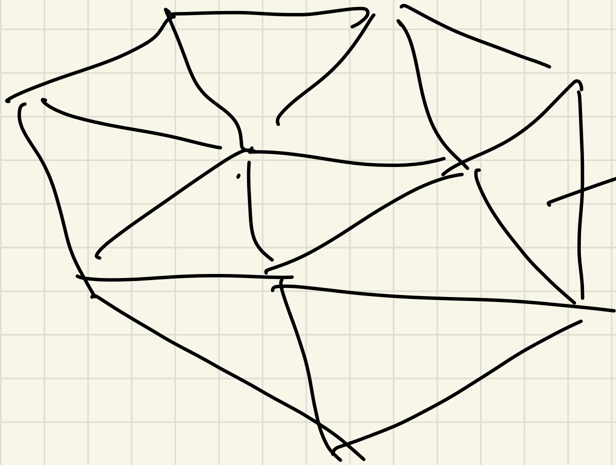
consequence:

$$\| u - u_h \|_1 \leq \frac{\Gamma}{\gamma} \inf_{v \in S_h} \| u - v \|_1$$

generalization: Céa lemma

finite elements

sequence of triangulations $\mathcal{T}_h$



$t \in \mathbb{R}^d$

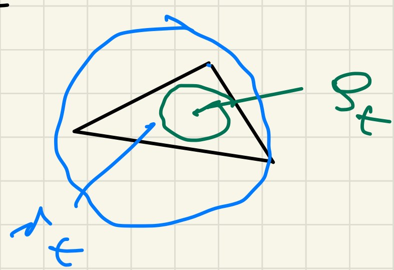
$t = \bar{t}$

assumption: $\partial\Omega$ polygonal

$$\bar{\Omega} = \bigcup_{t \in \mathcal{T}_h} t$$

$$h = \min_{t \in \mathcal{T}_h} \operatorname{diam}(t)$$

$$\sigma_h = \max_{t \in \mathcal{T}_h} \frac{r_t}{\rho_t} \le \text{const}$$



Theorem 3

Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$

Then

$$\| u - u_h \|_1 \leq \frac{\Gamma}{\gamma} \sigma_h \, h \, | u |_2$$

$$| u |_2^2 = \sum_{i,j=1}^{d} \| u_{x_i x_j} \|_0^2$$

Proof: $(d = 1, 2, 3)$

nodal interpolation:

$$I_h : H^2(\Omega) \longrightarrow S_h$$

$$\| u - u_h \|_1 \leq \frac{\Gamma}{\gamma} \| u - I_h u \|_1$$

$$\leq \frac{\Gamma}{\gamma} \sigma_h \, h \, | u |_2$$

highly - oscillating coefficients:

$$\alpha : \mathbb{R}^d \to \mathbb{R} \quad [0,1]^d \text{ periodic}$$

$$\alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right) \quad \varepsilon\text{- periodic } \varepsilon > 0$$

$$u_\varepsilon \in H : a_\varepsilon(u_\varepsilon, v) = l(v) \quad \forall v \in H$$

$$a_\varepsilon(u, w) = \int_\Omega \alpha_\varepsilon \nabla v \cdot \nabla w \, dx$$

$$\| u_\varepsilon - u_{\varepsilon,h} \|_1 \le C h \, | u_\varepsilon |_2$$

What happens for $\varepsilon \to 0$

example:

$\Omega = (0,1)$, $\alpha(y) = [2+\sin(2\pi y))^{-1}$

$f = -1$

$$-\left(\alpha\left(\frac{x}{\varepsilon}\right) u_\varepsilon'\right)' = -1 \text{ in } (0,1)$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 0$$

solution:

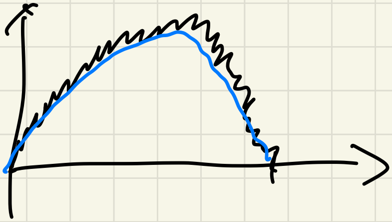
$$u_\varepsilon'(x) = (x+c)\left(2+\sin\left(2\pi \frac{x}{\varepsilon}\right)\right)$$

$$u_\varepsilon(x) = x(x+2c) + \left(\frac{\varepsilon}{2\pi}\right)^2 \sin\left(\frac{2\pi}{\varepsilon}x\right) + \frac{\varepsilon}{2\pi}\left(c-(x+c)\cos\left(\frac{2\pi}{\varepsilon}x\right)\right)$$

$$c = \frac{1}{2}\left(\frac{\varepsilon}{2\pi}-1\right)$$

order with respect to powers of ε:

$$u_\varepsilon(x) = x(x-1) + \frac{\varepsilon}{2\pi}\left(x-\frac{1}{2}\right)\left(1-\cos\left(\frac{2\pi}{\varepsilon}x\right)\right) + \left(\frac{\varepsilon}{2\pi}\right)^2\left(\sin\left(\frac{2\pi}{\varepsilon}x\right)-\frac{1}{2}\right)$$



$$\| u_\varepsilon \|_1 = \| u_\varepsilon' \|_0 \le \text{const}$$

$$u_\varepsilon''(x) = \Big((x + c)(2 + \sin(\tfrac{2\pi}{\varepsilon} x) \Big)'$$

$$= (2 + \sin(\tfrac{2\pi}{\varepsilon} x) +$$

$$(x + c) \tfrac{2\pi}{\varepsilon} \cos(\tfrac{2\pi}{\varepsilon} x)$$

$$| u_\varepsilon'' | \le C \, \varepsilon^{-1}$$

consequence:

$$\| u_\varepsilon - u_{\varepsilon,h} \|_1 \le C h \, | u_\varepsilon |_2 \le c \, \frac{h}{\varepsilon}$$

$$\frac{h}{\varepsilon} \to 0 \iff h = o(\varepsilon) \; .$$

i.e. resolution of oscillations by $\mathcal{T}_h$

way out:

- approximation of problem: oscillating coefficients

  $\leftrightarrow$ smooth coefficients

  $\leftrightarrow$ homogenization

  powerfull tool in mathematical modelling

- ε - depend. + discretization preserving $O(h)$ error bound independently of ε

  $\leftrightarrow$ multiscale finite elements