2. Random partial differential equations

2.1 Preliminaries

sample space Ω:
set of all possible outcomes of an experiment

event A: $A \subset \Omega$

σ-algebra $\mathcal{E}$: set of events with

(i) $\emptyset \in \mathcal{E}$, $\Omega \in \mathcal{E}$

(ii) $A \in \mathcal{E} \Rightarrow \Omega \setminus A \in \mathcal{E}$

(iii) $A_n \in \mathcal{E}, n \in \mathbb{N} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{E}$

Kolmogoroff

examples:

1) die: $\Omega = \{1, \dots, 6\}$

$\mathcal{E}_1 = 2^{\Omega}$ (set of all subsets)

$\mathcal{E}_2 = \{\emptyset, \{1, \dots, 6\}\}$

2) Borel sets: $Bor(\mathbb{R}) \subset 2^{\mathbb{R}}$

smallest σ-algebra such that all open intervals in $\mathbb{R}$ are contained in $Bor(\mathbb{R})$.

e.g. $(-\infty, x] \quad \forall x \in \mathbb{R}$

Note: $A \in Bor(\mathbb{R})$

$\not\Leftarrow$ A Lebesgue-measurable probability measure $P : \mathcal{E} \to \mathbb{R}$

with

(i) $P(\Omega) = 1$

(ii) $0 \leq P(A) \leq 1 \quad \forall A \in \mathcal{E}$

(iii) $A_n \in \mathcal{E}, \; A_n \cap A_m = \emptyset, \; n, m \in \mathbb{N}$

$$\Rightarrow P\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} P(A_n)$$

examples

1) chi : $P(A) = \#A / \#\Omega$

2) Borel sets :

$$P(A) = \int_A f(x)\, dx \qquad A \in \mathrm{Borel}(\mathbb{R})$$

with f integrable and

$f(x) \geq 0, \quad \int_\Omega f(x)\, dx = 1$

probability space: $(\Omega, \mathcal{E}, P)$

complete:

$A \in \mathcal{E}$, $P(A) = 0$ and $B \subset A$
$\Rightarrow B \in \mathcal{E}$

random variable: $v : \Omega \to \mathbb{R}$
$\mathcal{E}$ - measurable, i.e.

- all sections of v are $\mathcal{E}$-measurable

- $\{ \omega \in \Omega \mid v(\omega) \leq x \} \in \mathcal{E}$
  is measurable $\forall x \in \mathbb{R}$

- $v^{-1}(\mathrm{Bor}(\mathbb{R})) \in \mathcal{E}$

realization: $v(\omega_1)$ $\omega_1 \in \Omega$

samples: $v(\omega_1), \dots v(\omega_n)$
$\omega_i \in \Omega$

probability distribution

$$F_v : \mathbb{R} \to \mathbb{R}$$

$$F_v(x) = P(\{\omega \in \Omega \mid v(\omega) \leq x\})$$

$$=: P(v \leq x) \in \mathbb{R}$$

note: $P(x \leq v \leq y) = F_v(y) - F_v(x)$

expectation value:

$$E(v) = \int_\Omega v(\omega)\, dP$$

$$=: \lim_{\Delta x \to 0} \sum_{i \in \mathbb{Z}} x_i\, P(x_i \leq v \leq x_{i+1})$$

with $x_i = i \Delta x$, $i \in \mathbb{Z}$, $\Delta x > 0$

Remark:

$$E(v) = \lim_{\Delta x \to 0} \sum_{i \in \mathbb{Z}} x_i \left(F_v(x_{i+1}) - F_v(x_i) \right)$$

$$=: \int_{-\infty}^{\infty} x\, dF_v(x)$$

(Stieltjes integral)

Lemma 1:

$$E(\alpha v) = \alpha E(v) \quad \forall \alpha \in \mathbb{R}$$

probability density:

$$f_v := F_v' : \mathbb{R} \to \mathbb{R}$$

Remark: does not always exist.

$$\int_{-\infty}^{\infty} f_v(x)\,dx = F_v(\infty) - F_v(-\infty) = 1$$

$f(x) \geq 0$ as $F_v(x)$ is monotone

Lemma 2:

If v admits a density f_v, then

$$E(v) = \int_{-\infty}^{\infty} x\, f_v(x)\,dx$$

idea of proof:

$$E(\nu) = \lim_{\Delta x \to 0} \sum_{i \in \mathbb{Z}} x_i \cdot \left(F_\nu(x_{i+1}) - F_\nu(x_i) \right)$$

$$= \lim_{\Delta x \to 0} \sum_{i \in \mathbb{Z}} x_i \frac{1}{\Delta x} \left(F_\nu(x_{i+1}) - F_\nu(x_i) \right) \Delta x$$

$$= \int_{-\infty}^{\infty} x \, f_\nu(x) \, dx$$

examples:

(i) equidistribution: $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

(ii) Gaussian: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2} \right)$

(iii) exponential: $f(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

variance of v

$$\mathrm{Var}(v) = \mathbb{E}\left[(v - \mathbb{E}(v))^2 \right]$$

standard deviation of v:

$$\sigma(v) = \sqrt{\mathrm{Var}(v)}$$

example: Gaussian

$$\mathbb{E}(v) = m, \quad \mathrm{Var}(v) = \sigma^2$$

conditional probability

$$B \in \mathcal{E}, \quad P(B) > 0$$

restrict sample space from Ω to B:

new probability space $(B, B \cap \mathcal{E}, P(\cdot | B))$

$$P(A|B) := \frac{P(A \cap B)}{P(B)} \quad \text{(scaling)}$$

independent events:

$A, B \in \mathcal{E}$ independent $\overset{\text{Def.}}{\Longleftrightarrow}$

$$P(A \cap B) = P(A)\, P(B)$$

motivation: conditional probability

$$\frac{P(A \cap B)}{P(B)} = P(A|B) \overset{!}{=} P(A)$$

independent random variables:

v, w independent $\overset{\text{Def.}}{\Longleftrightarrow}$

all sections of v, w are indep.

joint probability distribution

$$F_{vw}(x,y) := P(v \le x,\ w \le y)$$

joint probability density

$$f_{vw}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{vw}(x,y)$$

Remarks

(i) $\displaystyle\int_{-\infty}^{\infty} f_{vw}(x,y)\,dy =$

$$\frac{\partial}{\partial x} \overbrace{F_{vw}(x,\infty)}^{=F_v(x)} - \frac{\partial}{\partial x} \overset{=0}{F_{vw}(x,-\infty)}$$

$$= \frac{\partial}{\partial x} F_v(x) = f_v(x)$$

(ii) v, w independent $\Rightarrow$

$$F_{vw}(x,y) = F_v(x)\, F_w(y)$$

$$f_{vw}(x,y) = f_v(x)\, f_w(y)$$

covariance of v, w:

$$\mathrm{Cov}(v,w) := E[(v-E(v))(w-E(w))]$$

v, w uncorrelated: $\mathrm{Cov}(v,w)$

Remark (exercise)

v, w independent $\Rightarrow$ v, w uncorrelated

v, w uncorrelated $\not\Rightarrow$ v, w independent

sums and products of random variables

$$(v+w)(\omega_1, \omega_2) = v(\omega_1) + w(\omega_2)$$

$$(v \cdot w)(\omega_1, \omega_2) = v(\omega_1) \cdot w(\omega_2)$$

Remark:

$$E(v+w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{vw}(x,y)\,dx\,dy$$

$$E(v \cdot w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y\, f_{vw}(x,y)\,dx\,dy$$

Lemma 3

(i) $E(v+w) = E(v) + E(w)$

(ii) v, w independent

$$E(v \cdot w) = E(v)\,E(w)$$

(iii) $\mathrm{Var}(\alpha v) = \alpha^2 \mathrm{Var}(v)$

(iv) v, w independent

$$\mathrm{Var}(v+w) = \mathrm{Var}(v) + \mathrm{Var}(w)$$

Proof:

(i) $E(v+w) =$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) f_{vw}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{vw}(x,y)\, dy \right) dx$$
$$+ \int_{-\infty}^{\infty} y \left(\int_{-\infty}^{\infty} f_{vw}(x,y)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} x\, f_v(x)\, dx + \int_{-\infty}^{\infty} y\, f_w(y)\, dy$$

$$= E(v) + E(w)$$

(ii) $E(vw) =$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y\, f_{vw}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y\, f_v(x)\, f_w(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_v(x)\, dx \int_{-\infty}^{\infty} y\, f_w(y)\, dy$$

$$= E(v)\, E(w)$$

(iv) $\mathrm{Var}(v+w)$

$$= E\left((v+w - E(v+w))^2 \right)$$

$$= E\left[((v - E(v)) - (w - E(w)))^2 \right]$$

$$= E\left[(v - E(v))^2 + (w - E(w))^2 \right.$$

$$\left. 2(v - E(v))(w - E(w)) \right]$$

$$= \mathrm{Var}(v) + \mathrm{Var}(w)$$

$$+ 2\,\overset{=0}{\mathrm{Cov}(v,w)}$$