Law of large numbers

Lemma 4 (Markov's lemma)

$$P(v \geq \alpha)\,\alpha \leq E(v) \quad \forall \alpha \in \mathbb{R}$$

Lemma 5 (Chebyshev's inequ.)

$$P(|v - E(v)| \geq k) \leq \frac{\operatorname{Var} v}{k^2} \quad \forall k \in \mathbb{N}$$

Proposition 6 (Law of large numbers)

Let v_i, $i = 1, \dots, M$, independent, identically distributed RVs and

$$w(\omega_1, \dots, \omega_M) = 1/M \sum_{i=1}^{M} v_i(\omega_i)$$

Then $E(w) = E(v_i)$, $i = 1, \dots, M$

and $\forall \varepsilon > 0$ there is $c_\varepsilon > 0$:

$$P(|w - E(w)| \leq c_\varepsilon M^{-1/2}) \leq 1 - \varepsilon$$

Monte-Carlo Method

Approximate $E(v)$ by sampling of w:

Select independent samples $v(\omega_i)$ of v

and compute

$$w(\omega_1, \dots, \omega_M) = 1/M \sum_{i=1}^{M} v(\omega_i)$$

Then

$$|w(\omega_1, \dots, \omega_M) - E(v)| \leq C_\varepsilon M^{-1/2}$$

with probability $1-\varepsilon$

2.2 Random fields

probability space (Ω, Σ, P)

lin. space of random variables V:

$$(\alpha v + \beta w)(\omega_1, \omega_2) = \alpha v(\omega_1) + \beta w(\omega_2)$$

$1 \leq p < \infty$:

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(\omega)|^p \, dP(\omega) \right)^{1/p} \in \mathbb{R} \cup \{+\infty\}$$

$p = \infty$:

$$\|v\|_{L^\infty(\Omega)} = \inf \{ s \in \mathbb{R} \mid P(|v(\omega)| \leq s) = 1 \}$$

Remark:

v P-measurable $\Longleftrightarrow$

i.e. $v^{-1}(I) \in \Sigma \quad \forall \, I \in \mathrm{Bor}(\mathbb{R})$

$$L^p(\Omega) = \{ v : \Omega \to \mathbb{R} \mid P\text{-measurable}, \; \|v\|_{L^p(\Omega)} < \infty \} \quad 1 \leq p \leq \infty$$

norms on subspaces of V

Remark:

$L^p(\Omega)$, $1 \le p \le \infty$, are Banach spaces

$L^2(\Omega)$ is a Hilbert space with scalar product

$$(v, w)_{L^2(\Omega)} = E(v \cdot w)$$

$$= \int_{\Omega \times \Omega} v(\omega_1)\, w(\omega_2)\, dP(\omega_1, \omega_2)$$

$L^2(\Omega)$: second order random variables

vector-valued random variables

$$(v_i)_{i=1,\dots,n}$$

parametrized random variable

discrete parameter : $i = 1, \dots, n$

random fields

spatial domain $D \subset \mathbb{R}^d$

(polygonal, convex)

parametrized collection of RVs:

$$\{ v_x \}_{x \in D} = v(\cdot, x), \quad x \in D$$

- random variable for each $x \in D$:

$$v(\cdot, x)$$

- spatial function for each $\omega \in \Omega$:

$$v(\omega, \cdot)$$

Remark: stochastic process

$$\{ v_t \}_{t \geq 0} = v_t, \quad t \geq 0$$

tensor-product character

motivation: finite dimensions

$H_1 = \mathbb{R}^n$ + discrete parametrization:

- element of $\mathbb{R}^n$ for each $i = 1, \dots, m$

$$W_1 = \{ u = (u_i)_{i=1}^m \mid u_i \in \mathbb{R}^n \}$$

$H_2 = \mathbb{R}^m$ + discrete parametrization:

- element of $\mathbb{R}^m$ for each $j = 1, \dots, n$

$$W_2 = \{ u = (u_j)_{j=1}^n \mid u_j \in \mathbb{R}^m \}$$

tensor product space

$$W_3 = H_1 \otimes H_2$$

$$= \operatorname{span} \{ u \mid u = (v_i w_j) \in \mathbb{R}^{m \times n}$$

$$W_1 \cong W_2 \cong W_3$$

goal: similar construction for

$$H_1 = L^2(\Omega), \quad H_2 = H = H_0^1(D)$$

space of random fields $v : \Omega \to H$

σ-algebra in H: $\quad Bor(H) \subset 2^H$

smallest σ-algebra that contains open subsets of H

$v : \Omega \to H$ measurable $\iff$

$$v^{-1}(I) \in \Sigma \quad \forall I \in Bor(H)$$

$$\|v\|_{L^2(\Omega, P, H)} = \left(\int_\Omega \|v(\omega, \cdot)\|_H^2 \, dP(\omega) \right)^{1/2}$$

$$W_1 = L^2(\Omega, P, H)$$
$$= \{ v : \Omega \to H \mid v \text{ measurable}, \|v\|_{L^2(\Omega, P, H)} < \infty \}$$

$W_2 = \{ D \to L^2(\Omega) \}$ is not interesting tensor-product space:

$$W_3 = L^2(\Omega) \otimes H$$
$$:= \text{ completion of } V,$$

$$V = \text{span}\{ u : \Omega \times D \to \mathbb{R} \mid u(\omega, x) = v(\omega)\, w(x),\ v \in L^2(\Omega), w \in H\}$$

w.r.t. linear extension (span!) of the scalar product

$$(v_1 w_1, v_2 w_2)_V := (v_1, v_2)_{L^2(\Omega)} (w_1, w_2)_H$$

Proposition 8

$$L^2(\Omega, P, H) \cong L^2(\Omega) \otimes H$$