tensor-product character of random fields

space of random fields $v : \Omega \to H$:

σ-algebra in H: $\mathcal{B}or(H) \subset 2^H$

(smallest σ-algebra that contains open subsets of H)

$v : \Omega \to H$ measurable $\iff$

$$v^{-1}(I) \in \Sigma \quad \forall I \in \mathcal{B}or(H)$$

scalar product:

$$(v, w)_{L^2(\Omega, P, H)} = \int_\Omega (v, w)_H \, dP$$

Hilbert space: $W_1 := L^2(\Omega, P, H)$

$$= \{ v : \Omega \to H \mid v \text{ measurable}, \; \|v\|_{L^2(\Omega, P, H)} < \infty \}$$

tensor-product space:

- scalar product on products:

$$v(\omega,x) = v^1(\omega)v^2(x), \quad w(\omega,x) = w^1(\omega)w^2(x)$$

$$(v^1v^2, w^1w^2)_V = (v^1, w^1)_{L^2(\Omega)}(v^2, w^2)_H$$

- linear extension:

$$V = \text{span}\{u : \Omega \times D \to \mathbb{R} \mid$$
$$u(\omega,x) = u^1(\omega)\,u^2(x), \; (\omega,x) \in \Omega \times D$$

$$(v, w)_V = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (v_i^1, w_j^1)_{L^2(\Omega)} (v_i^2, w_j^2)_H$$

- completion:

$$W_3 = L^2(\Omega) \otimes H := \text{completion of } V$$

$$\text{w.r.t.} \quad \|v\|_V = (v,v)_V^{1/2}$$

Proposition 8

$$L^2(\Omega, P, H) \cong L^2(\Omega) \otimes H$$

Proof:

a) $(v, w)_V = (v, w)_{L^2(\Omega, P, H)} \quad \forall v, w \in V$

$$v = \sum_{i=1}^{n} \alpha_i v_i^1(\omega) v_i^2(x)$$

$$w = \sum_{j=1}^{m} \beta_j w_j^1(\omega) w_j^2(x)$$

$$(v, w)_{L^2(\Omega, P, H)} = \int_\Omega (v(\omega), w(\omega))_H \, dP$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \int_\Omega (v_i^1(\omega) v_i^2, w_j^1(\omega) w_j^2)_H \, dP$$

$$= \sum \sum \alpha_i \beta_j (v_i^2, w_j^2)_H \int_\Omega v_i^1(\omega) w_j^1(\omega) \, dP$$

$$= \sum \sum \alpha_i \alpha_j (v_i^1, w_j^1)_{L^2(\Omega)} (v_i^2, w_j^2)_H$$

$$= (v, w)_V$$

b) $V \subset L^2(\Omega, P, H)$ dense

a) $\Rightarrow \|u\|_V = \|u\|_{L^2(\Omega, P, H)} < \infty$

$\Rightarrow V \subset L^2(\Omega, P, H)$

$H = H_0^1(D)$ separable

$\Rightarrow \exists$ ONB $\{e_i\}_{i \in \mathbb{N}}$ of H

Let $g \in L^2(\Omega, P, H), g(\omega) \in H \ \forall \omega \in \Omega$

$g_i(\omega) = (g(\omega), e_i)_H \Rightarrow g_i \in L^2(\Omega)$

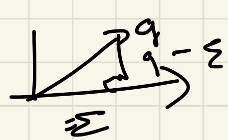
$\Rightarrow \sum_{i=1}^{n} g_i e_i \in V$

(i) $f(\omega) := \| g(\omega) - \sum_{i=1}^{n} g_i(\omega) e_i \|_H^2 \to 0$

(ii) $f_n(\omega) \leq \| g(\omega) \|_H^2$

as $\| g(\omega) - \sum_{i=1}^{n} g_i(\omega) e_i \|^2 + \| \sum_{i=1}^{n} g_i(\omega) e_i \|_H^2 = \| g(\omega) \|_H^2$

(iii) $f \equiv 0$ measurable



Lebesgue's dominated convergence

$$\| g - \sum_{i=1}^{n} g_i e_i \|^2_{L^2(\Omega, P, H)}$$

$$= \int_{\Omega} f_n(\omega)\, dP(\omega) \longrightarrow 0 \quad \text{as } n \to \infty$$

$\Leftrightarrow$ $L^2(\Omega, P, H)$ is a completion of V which is unique up to isomorphisms. $\blacksquare$

2.3 Random elliptic pdes

Find $u \in X = L^2(\Omega, P, H)$

such that $(*)$:

$$\underbrace{\int_\Omega \int_D K(\omega,x) \nabla u(\omega,x) \nabla v(\omega,x)\, dx\, dP}_{a(u,v)}$$

$$= \underbrace{\int_\Omega \int_D f(\omega,x) v(\omega,x)\, dx\, dP}_{\ell(v)}$$

$$\forall v \in X$$

$\Longleftrightarrow$

$$\int_D K(\omega,x) \nabla u(\omega,x) \nabla v(\omega,x)\, dx$$

$$= \int_D f(\omega,x) v(\omega,x)\, dx$$

$$\forall v \in X$$

almost surely, a.s., in Ω

Proposition 9

Let

(i) $f \in L^2(\Omega, P, L^2(D))$

(ii) $0 < K_{min} \leq K(\omega, x) \leq K_{max} < \infty$ a.s. in $\Omega \times D$

Then $(*)$ has a unique solution.

Proof:

$$|\ell(v)| = \int_\Omega (f(\omega_1)\, v(\omega_2))_{L^2(D)}\, dP(\omega_1, \omega_2)$$

$$\leq \int_\Omega \| f(\omega_1)\|_{L^2(D)} \| v(\omega_2)\|_{L^2(D)}\, dP(\omega_1, \omega_2)$$

$$\leq \| f \|_{L^2(\Omega, P, L^2(D))} \| v \|_X$$

$$|a(v, w)| = \int_\Omega (K \nabla v, \nabla w)_{L^2(D)}\, dP$$

$$\leq K_{max} \int_\Omega (v, w)_H\, dP$$

$$\leq K_{max} \| v \|_X \| w \|_X$$

$$a(v, v) \geq K_{min} \| v \|_X \quad \text{(Poincaré)}$$

2. 4. Monte-Carlo finite elements

semi-discretization in space:

triangulation $\mathcal{T}_h$ of D, nodes $\mathcal{N}_h$
shape regular
finite elements $S_h \subset H = H_0^1(D)$
nodal basis λ_p, $p \in \mathcal{N}_h$

subspace (infinite-dimensional)

$$X_h = \Big\{ v : \Omega \times D \to \mathbb{R} \;\Big|\; v(\omega, x) = \sum_{p \in \mathcal{N}_h} v_p(\omega)\, \lambda_p(x), \quad v_p \in L^2(\Omega) \Big\}$$

Remark : X_h closed subspace of X

Proof :

X_h is isomorphic with $L^2(\Omega)^N$, $N = |N_h|$

i.e. $L^2(\Omega)^N \ni \alpha = (\alpha_p)$

$$\longrightarrow \quad F(\alpha) = \sum_{p \in N_h} \alpha_p \lambda_p \in X_h$$

is bijective and

$$c \, \| F(\alpha) \|_X^2 \leq E[\langle \alpha, \alpha \rangle] \leq C \, \| F(\alpha) \|_X^2$$

$$\| F(\alpha) \|_X^2 = \sum_{p,q \in N_h} \int_\Omega (\alpha_p \lambda_p, \alpha_q \lambda_q)_H \, dP$$

$$= \sum_{p,q \in N_h} \underbrace{(\lambda_p, \lambda_q)_H}_{a_{pq}} (\alpha_p, \alpha_q)_{L^2(\Omega)}$$

$$= E[\langle A \alpha, \alpha \rangle] \leq \lambda_{max}^{(A)} E[\langle \alpha, \alpha \rangle]$$

Ritz - Galerkin method

$$u_h \in X_h : \quad a(u_h, v) = \ell(v) \quad \forall v \in X_h$$

Céa lemma:

$$\| u - u_h \|_X \leq C \inf_{v \in X_h} \| u - v \|_X$$

Remark: $\inf_{v \in X_h} \| u - v \|_X \to 0$ as $h \to 0$

Proposition 10

Let $u \in L^2(\Omega, P, H^2(D) \cap H)$

Then there is a $C > 0$ such that

$$\| u - u_h \|_X \leq C h \| u \|_{L^2(\Omega, P, H^2(D) \cap H)}$$

Proof:

a) Let $\omega \in \Omega$

Then $u(\omega,\cdot) \in H_0^1(D) \cap H^2(D)$

nodal interpolation $I_h : H^2(D) \to S_h$

$$\| u(\omega,\cdot) - I_h u(\omega,\cdot) \|_{H^1(D)}$$

$$\leq c\, h \, \| u(\omega,\cdot) \|_{H^2(D)}$$

b) $\| u - I_h u \|_X =$

$$\left(\int_\Omega \| u(\omega,\cdot) - I_h u(\omega,\cdot) \|^2_{H^1(D)} \, dP(\omega) \right)^{1/2}$$

$$\leq c\, h \int_\Omega \| u(\omega,\cdot) \|^2_{H^2(D)} \, dP(\omega))^{1/2}$$

$$= c\, h \, \| u \|_{L^2(\Omega, P, H^2(D))}$$

Problem: u_h is not computable

polynomial chaos:

- polynomial Galerkin approximation of coefficients $\alpha_p \in L^2(\Omega)$

Monte-Carlo approximation

of expectation $\mathbb{E}[u_h] = \int_\Omega u_h \, dP$

Monte Carlo method

for $i = 1, \dots, M$

- sample realizations $K(\cdot, \omega_j)$, $f(\cdot, \omega_j)$
- compute $u_h(\omega_j) \in S_h$ from

$$\int_D K(\omega_j, \cdot) \nabla u_h \nabla v \, dx = \int_D f(\omega_j, \cdot) v \, dx \quad \forall v \in S_h$$

- average in S_h

$$E_M[u_h](\omega_1, \dots, \omega_M) = \frac{1}{M} \sum_{j=1}^{M} u_h(\omega_j).$$

goal: error estimates

a) expectation sense (law of large numbers)

$$\| E[u] - E_M[u_h] \|_X \le C (M^{-1/2} + h)$$

b) pointwise sense (central limit theorem)

$$\| E[u] - E_M[u_h] \|_H (\omega_1, \dots, \omega_M) \le C (M^{-\alpha} + h^{\beta})$$

almost surely