random elliptic problem

Find $u \in X = L^2(\Omega, P, H)$
such that $(*)$

$$a(u, v) = \ell(v) \quad \forall\, v \in X$$

with

$$a(v, w) = E\left[(K \nabla v, \nabla w)_{L^2(D)} \right]$$

$$\ell(v) = E\left[(f, v)_{L^2(D)} \right]$$

(i) $0 < K_0 \leq K(\omega, x) \leq K_1 < \infty$ a.s. $\forall x \in D$

(ii) $f \in L^2(\Omega, P, L^2(D))$

Existence & uniqueness (Lax-Milgram)

Monte-Carlo finite elements

semi-discretization in space:

triangulation $\mathcal{T}_h$ of D, nodes $\mathcal{N}_h$

shape regular

finite elements $S_h \subset H = H_0^1(D)$

nodal basis λ_p, $p \in \mathcal{N}_h$

closed subspace

$$X_h = \{ v : \Omega \times D \to \mathbb{R} \mid$$

$$v(\omega, x) = \sum_{p \in \mathcal{N}_h} v_p(\omega) \lambda_p(x)$$

$$v_p \in L^2(\Omega) \}$$

Ritz - Galerkin method

$$u_h \in X_h : \quad a(u_h, v) = \ell(v) \quad \forall v \in X_h$$

Céa lemma:

$$\| u - u_h \|_X \leq C \inf_{v \in X_h} \| u - v \|_X$$

Remark: $\inf_{v \in X_h} \| u - v \|_X \to 0$ as $h \to 0$

Proposition 10

Let $u \in L^2(\Omega, P, H^2(D) \cap H)$

Then there is a $C > 0$ such that

$$\| u - u_h \|_X \leq C h \| u \|_{L^2(\Omega, P, H^2(D) \cap H)}$$

Monte Carlo method

for $i = 1, \ldots, M$

- sample realizations $K(\cdot\, \omega_j), f(\cdot, \omega_j)$
- compute $u_h(\omega_j) \in S_h$ from

$$\int_D K(\omega_j, \cdot) \nabla u_h \nabla v \, dx = \int_D f(\omega_j, \cdot) v \, dx \quad \forall v \in S_h$$

- average in S_h

$$E_M[u_h](\omega_1, \ldots, \omega_M) = \frac{1}{M} \sum_{j=1}^{M} u_h(\omega_j).$$

goal: error estimates

a) expectation sense (law of large numbers)

$$\| E[u] - E_M[u_h] \|_X \le C(M^{-1/2} + h)$$

b) pointwise sense (central limit theorem)

$$\| E[u] - E_M[u_h] \|_H (\omega_1, \ldots, \omega_M) \le C(M^{-\alpha} + h^\beta)$$

almost surely

separation of error contributions:

$$E[u] - E_M[u_h]$$
$$= E[u] - E[u_h]$$
$$+ E[u_h] - E_M[u_h]$$

Proposition 11 (spatial error)

Let $u \in L^2(\Omega, P, H^2(D))$

$$\| E[u] - E[u_h] \|_X$$
$$\leq c\, h \, \| u \|_{L^2(\Omega, P, H^2(D))}$$

Proof:

a) $\| E[v] \|^2_{H^1(D)} \leq E[\| v \|^2_{H^1(D)}] \; \forall v \in X$

$$\| E[v] \|^2_{H^1(D)} = \int_D \Big(\int_\Omega 1 \cdot v(\omega, x) \, dP \Big)^2 dx$$

$$+ \int_D \Big| \nabla_x \int_\Omega v(\omega, x) \, dP \Big|^2 dx$$

$$\overset{C.S.}{\leq} \int_D \Big(\int_\Omega v(\omega, x)^2 \, dP \int_\Omega 1^2 \, dP \Big) dx$$

$$+ \int_D \sum_{k=1}^{d} \Big(\int_\Omega 1 \cdot \partial_{x_k} v(\omega, x) \, dP \Big)^2 dx$$

$$\leq \int_\Omega \Big(\int_D v^2 + |\nabla v|^2 \, dx \Big) dP$$

$$= E[\| v \|^2_{H^1(D)}]$$

b) $\| E[u] - E[u_h] \|^2_{H^1(D)} = \| E[u - u_h] \|^2_{H^1(D)}$

$$\overset{a)}{\leq} E[\| u - u_h \|^2_{H^1(D)}] = \| u - u_h \|^2_X$$

$$\overset{\text{Prop. 10}}{\leq} c \, h^2 \| u \|^2_{L^2(\Omega, P, H^2(D))}$$

stochastic error :

$$\mathbb{E}[u_h] - E_M[u_h]$$

in expectation :

Proposition 12 (stochastic error)

$$\| \mathbb{E}[u_h] - E_M[u_h] \|_X \leq C M^{-1/2} \|u\|_X$$

Proof :

a) $\| \mathbb{E}[u_h] - E_M[u_h] \|_X$

$$\leq M^{-1/2} \underbrace{\| \mathbb{E}[u_h] - u_h \|_X}_{= \sigma(u_h) = \mathrm{Var}(u_h)^{1/2}}$$

Let $\mathbb{E}[u_h] = 0$, else set $w_h = u_h - \mathbb{E}[u_h]$

$$\| E_M [u_h] \|_X^2 = E[\, \| E_M [u_h] \|_{H^1}^2]$$

$$= M^{-2} E\Big[\sum_{i=1}^{M} \| u_{h,i} \|_{H^1}^2$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{M} (u_{h,i} , u_{h,j})_H \Big]$$

indep.

$$= M^{-2} \sum_{i=1}^{M} \underbrace{E[\| u_{h,i} \|_{H^1}^2]}_{= E[\| u_h \|_H^2]}$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{M} \underbrace{(E\, u_{h,i} , E\, u_{h,j})_H}_{=0}$$

$$= M^{-2} M \, E[\| u_h \|_H^2]$$

$$= M^{-1} \| u_h \|_X^2 = M^{-1} \| E[u_h] - u_h \|_X$$

b) $\| E[u_n] - u_n \|_X \leq \| u_n \|_X$

$$\| E[u_n] - u_n \|_X^2 = E\left[\| E[u_n] - u_n \|_{H^1}^2 \right]$$

$$= E\left[\| E[u_n] \|_{H^1}^2 - 2 \left(E[u_n], u_n \right)_{H^1} + \| u_n \|_{H^1}^2 \right]$$

$$= \| E[u_n] \|_{H^1}^2 - 2 \| E[u_n] \|_H^2 + \| u_n \|_X^2$$

$$= \| u_n \|_X^2 - \| E[u_n] \|_{H^1}^2 \leq \| u_n \|_X^2$$

c) $\| u_n \|_X \leq c \| u \|_X$

$$\| u_n \|_X^2 \leq \frac{1}{K_1} a(u_n, u_n) = \frac{1}{K_1} \ell(u_n)$$

$$= \frac{1}{K_1} a(u, u_n) \leq \frac{K_2}{K_1} \| u \|_X \| u_n \|_X$$

$$c = \frac{K_2}{K_1}$$

$\boxed{M}$

Theorem 13

Let $u \in L^2(\Omega, P, H^2(D))$

Then

$$\| E[u] - E_M[u_h] \|_X$$

$$\leq c\, h \, \| u \|_{L^2(\Omega, P, H^2(D))} + c\, M^{-1/2} \| u \|_X$$

$$\leq c\, (h + M^{-1/2}) \| u \|_{L^2(\Omega, P, H^2(D))}$$

efficiency:

equilibrate h_j and $M^{-1/2}$

pointwise error

Theorem 14 (pointwise error

Let $u \in L^2(\Omega, P, H^2(D))$ and $M_k = 2^k$. Then

$$\| E[u] - E_{M_k}[u_h] \|_{H^1(\omega_1, \dots, \omega_{M_k})}$$

$$\leq C \|u\|_{L^2(\Omega, P, H^2(D))} h$$

$$+ C_\alpha M_k^{-\alpha} \quad , \quad \alpha \in (0, 1/2)$$

holds for almost all samples $u(\omega_i)$, $i = 1, \dots, M_k$

Multilevel Monte Carlo methods

computational cost:

$$err_{h,M} = \| E[u] - E_M[u_h] \|_X \le C(h + M^{-1/2})$$

goal: $err_{h,M} = O(h)$

at minimal computational cost

Assumption MG

The FE-solution $u_h(\cdot, \omega_j)$ upto $O(h_j)$ can be computed at optimal comp. cost $O(n_h)$, $n_h = \# \mathcal{N}_h$

Remark: Can be fulfilled by MG

computational cost of Monte Carlo FEM

Set $M_h = n_h^{2/d} = \frac{1}{h^2}$

$(n_h \approx h^{-d} \, , \; h \approx n_h^{-1/d})$

— $err_h = O(h + n_h^{-1/d}) = O(h)$

— $work_h = n_h \cdot n_h^{2/d}$

$$= n_h^{1+2/d} = \begin{cases} n_h^3 & d=1 \\ n_h^2 & d=2 \\ n_h^{5/3} & d=3 \end{cases}$$

Proposition 16

Assume (MG). Then $err_h = O(h)$ can be achieved at $work = O(n_h^{1+2/d})$

Can we do better?

Assumption (H)

hierarchy of triangulations T_k, $k=0,\dots,j$

mesh size $h_k = c\,2^{-k}$

nodes N_k : $n_k = q\, n_{k-1}$, $q = 2^d$

telescope sum

$$E[u_j] = \sum_{k=0}^{j} E[u_k - u_{k-1}]$$

$$E[u_{-1}] := 0$$

multilevel approximation

$$E_j[u_j] = \sum_{k=0}^{j} E_{M_k}[u_k - u_{k-1}]$$

Lemma 17

Let $u \in L^2(\Omega, P, H^2(D))$. Then

$$\| E[u] - E_j[u_j] \|_X \leq c \left(h_j + \sum_{k=0}^{j} M_k^{-1/2} h_k \right)$$

Proof:

$$\| E[u] - E[u_j] \|_X \leq c\, h_j$$

$$\| E[u_j] - E_j[u_j] \|_X$$

$$\leq \sum_{k=0}^{j} \| E[u_k - u_k] - E_{M_k}[u_k - u_{k-1}] \|_X$$

$$\leq c \sum_{k=1}^{j} M_k^{-1/2} \| u_k - u_{k-1} \|_X$$

$$\| u_k - u_{k-1} \|_X \leq \| u_k - u \|_X + \| u - u_{k-1} \|_X$$

$$\leq c\, h_k$$

$\boxtimes$

choose:
$$M_k = (k+1)^{2(1+\varepsilon)} \left(\frac{h_k}{h_j}\right)^2 \quad \varepsilon > 0$$

accuracy:
$$\sum_{k=0}^{j} M_k^{-1/2} h_k = \sum_{k=0}^{j} (k+1)^{-(1+\varepsilon)} \frac{h_j}{h_k} h_k$$
$$= h_j \sum_{k=0}^{j} (k+1)^{-(1+\varepsilon)} \leq c\, h_j$$

work:
$$\operatorname{work}(E_j[u_j]) = \sum_{k=0}^{j} M_k (n_k + n_{k-1})$$
$$c \sum_{k=0}^{j} (k+1)^{2(1+\varepsilon)} 4^{j-k} \underbrace{q^k q^{j-k}}_{\approx n_j} q^{k-j}$$
$$= c\, n_j \sum_{k=0}^{j} (k+1)^{2(1+\varepsilon)} \left(\frac{4}{q}\right)^{j-k}$$
$$= c\, n_j (j+1)^{2(1+\varepsilon)} \sum_{k=0}^{j} \left(\frac{4}{2^d}\right)^{j-k}$$
$$\leq c\, n_j \begin{cases} j^{3+2\varepsilon} & d = 2 \\ j^{2+2\varepsilon} & d \geq 3 \end{cases}$$

Theorem 18

Let $d \geq 2$ and $u \in L^2(\Omega, P, H^2(D))$

Set $E_j[u_j] = \sum_{k=0}^{j} E_{M_k}[u_k - u_{k-1}]$

with $E_{M_0}[u_{-1}] := 0$ and

$$M_k = (k+1)^{2(1+\varepsilon)} 4^{j-k}, \quad \varepsilon > 0$$

Then

$$\| E[u] - E_j[u_j] \|_X = O(h_j)$$

and

$$\text{work}(E_j[u_j]) \leq c\, n_j \begin{cases} j^{3+2\varepsilon} & d=2 \\ j^{2+2\varepsilon} & d \geq 3 \end{cases}$$

optimal efficiency
of a deterministic problem
up to constants and
logarithmic terms