1.2.1 Multiscale asymptotics

$$-\operatorname{div}(\alpha_\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega \subset \mathbb{R}^d$$

$$u|_{\partial\Omega} = 0$$

$$\alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right)$$

$$\alpha : \mathbb{R}^d \to \mathbb{R}^d \qquad Y = [0,1]^d \text{ periodic}$$

example:

$$\Omega = (0,1), \quad \alpha(y) = \left(2 + \cancel{\sin(2\pi y)}\right)^{-1}$$

$$f = -1$$

$$-\left(\alpha\left(\frac{x}{\varepsilon}\right) u_\varepsilon'\right)' = -1$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 0$$

solution:

$$u_\varepsilon(x) = x(x-1) + \frac{\varepsilon}{2\pi}\left(x - \tfrac{1}{2}\right)\left(1 - \cos\left(\frac{2\pi}{\varepsilon} x\right)\right) + \frac{\varepsilon^2}{(2\pi)^2}\left(\sin\left(\frac{2\pi}{\varepsilon}\right) - \frac{1}{2}\right)$$

ansatz:

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots$$

$$= u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \dots$$

$y := \frac{x}{\varepsilon}$ $\qquad \nabla = \nabla_x + \frac{1}{\varepsilon} \nabla_y$

pde: $-\nabla \cdot (\alpha(y) \nabla u_\varepsilon(x)) = f(x)$

insert ansatz:

$$f(x) = \varepsilon^{-2} \nabla_y \cdot (\alpha(y) \nabla_y u_0(x,y))$$

insert ansatz:

$$
\begin{aligned}
-f(x) = {} & \varepsilon^{-2}\, \nabla_y \cdot \big(\alpha(y) \nabla_y u_0(x,y)\big) \\
& + \varepsilon^{-1} \Big(\nabla_y \cdot \big(\alpha(y) \nabla_x u_0(x,y)\big) \\
& \qquad + \nabla_x \cdot \big(\alpha(y) \nabla_y u_0(x,y)\big) \\
& \qquad + \nabla_y \cdot \big(\alpha(y) \nabla_y u_1(x,y)\big) \Big) \\
& + \varepsilon^{0} \Big(\alpha(y)\, \Delta_x u_0(x,y) \\
& \qquad + \nabla_y \cdot \big(\alpha(y) \nabla_x u_1(x,y)\big) \\
& \qquad + \nabla_x \cdot \big(\alpha(y) \nabla_y u_1(x,y)\big) \\
& \qquad + \nabla_y \cdot \big(\alpha(y) \nabla_y u_2(x,y)\big) \Big) \\
& + \varepsilon^{+1} \Big(\big(\alpha(y)\, \Delta_x u_1(x,y) \big) \\
& \qquad + \nabla_y \cdot \big(\alpha(y) \nabla_x u_2(x,y)\big) \\
& \qquad + \alpha(y)\, \nabla_x \cdot \big(\nabla_y u_2(x,y)\big) \\
& \qquad + \nabla_y \cdot \big(\alpha(y) \nabla_y u_3(x,y)\big) \Big) \\
& + \dots
\end{aligned}
$$

Lemma

$w : \mathbb{R}^d \to \mathbb{R}$ Y-periodic smooth

(i) ∇w Y-periodic

$$\int_Y w_{y_i} \, dy = 0$$

(ii) $$\int_Y \nabla_y \cdot (\alpha \nabla_y w) v \, dy$$

$$= - \int_Y \alpha \nabla_y w \cdot \nabla v \, dy$$

$\forall v$ Y-periodic

(iii) $$\int_Y \nabla_y \cdot (\alpha \nabla_y w) \quad dy = 0$$

(iv) $$- \nabla_y \cdot (\alpha \nabla_y w) = g$$

$\int_Y g \, dy = 0$ necessary cond. for existence

uniqueness up to constants.

$\underline{\varepsilon^{-2}}$: $\quad \nabla_y \cdot \left(\alpha(y) \nabla_y u_0(x,y) \right) \overset{!}{=} 0$

$$\Leftrightarrow \quad 0 = \int_Y \nabla_y \cdot \left(\alpha(y) \nabla_y u_0(x,y) \right) u_0(x,y) \, dy$$

$$= \int_Y \alpha(y) \, |\nabla u_0(x,y)|^2 \, dy$$

$$\Leftrightarrow \quad \nabla_y u_0(x,y) = 0 \quad \forall x \in \Omega$$

$$u_0(x,y) =: u(x)$$

$\underline{\varepsilon^{-1}}$:

$$0 = \nabla_y \cdot (\alpha \nabla_x u_0) + \cancel{\alpha \nabla_x \cdot (\nabla_y u_0)} + \nabla_y \cdot (\alpha \nabla_y u_1)$$

$$= \nabla_y \cdot \left[\alpha (\nabla_x u + \nabla_y u_1) \right]$$

ansatz for u_1:

$$\text{(Ansatz)} \quad u_1(x,y) = \sum_{i=1}^{d} u_{x_i}(x)\, w_i(y)$$

$$= \nabla_x u(x) \cdot w(y)$$

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} \in \mathbb{R}^d \qquad \nabla_y w = \begin{pmatrix} \nabla w_1 \\ \vdots \\ \nabla w_d \end{pmatrix} \in \mathbb{R}^{d \times d}$$

$$0 = \nabla_y \cdot \left(\alpha \left(\nabla_x u + \nabla_y (\nabla_x u \cdot w) \right) \right)$$

$$= \nabla_y \cdot \left(\alpha \left(\nabla_x u + \nabla_x u \, \nabla_y w \right) \right)$$

$$= \nabla_y \cdot \left(\alpha \nabla_x u \left(I + \nabla_y w \right) \right)$$

$$= \nabla_y \cdot \left(\alpha \sum_{i=1}^{d} u_{x_i} \left(e_i + \nabla_y w_i \right) \right)$$

$$= \nabla_x u \cdot \left(\nabla_y \cdot \left(\alpha \left(I + \nabla_y w \right) \right) \right)$$

sufficient: cell problem

$$\nabla_y \cdot \left(\alpha \left(I + \nabla_y w \right) \right) = 0 \quad y \in Y$$

componentwise cell problem

$$\nabla_y \cdot (\alpha(\nabla_y w_i + e_i)) = 0$$

$$(2) \quad -\nabla_y \cdot (\alpha \nabla_y w_i) = \nabla_y \cdot (\alpha e_i)$$

$$= \alpha_{y_i}$$

existence + uniqueness up to const.

$\underline{\varepsilon^0:}$

$$-f = \alpha_\varepsilon \Delta_x u$$

$$+ \nabla_y \cdot (\alpha \nabla_x u_1) + \alpha \nabla_x \cdot (\nabla_y u_1)$$

$$+ \nabla_y \cdot (\alpha \nabla_y u_2)$$

$$-\nabla_y (\alpha \nabla_2 u_2) = f$$

$$+ \nabla_y \cdot (\alpha (\nabla_x u_1))$$

$$+ \nabla_x (\alpha (\nabla_y u_1 + \nabla_x u))$$

$$-\nabla_y(\alpha \nabla_2 u_2) = f$$
$$+ \nabla_y \cdot (\alpha (\nabla_x u_1))$$
$$+ \nabla_x (\alpha (\nabla_y u_1 + \nabla_x u))$$

$$0 = \int_Y f + \nabla_y \cdot (\alpha (\nabla_x u_1)) + \nabla_x (\alpha (\nabla_y u_1 + \nabla_x u)) dy$$

(K)

eliminate u_1

using (Ansatz) and (Z)

homogenized pde :

$$-\operatorname{div}(\alpha^* \nabla u) = f \text{ in } \Omega$$

$$\alpha^* = (\alpha_{ij})_{i,j=1}^d$$

$$\alpha_{ij} = \int_Y \alpha (\nabla_y w_i + e_i) \cdot (\nabla w_j + e_j) dy$$

proof for $d=1$:

$$(\text{Ansatz})_{d=1} \qquad u_1 = u_x w$$

$$(Z)_{d=1} \qquad -(\alpha w_y)_y = \alpha_y$$

$$(K)_{d=1} \; 0 = \int_Y f + (\alpha u_{1,x})_y$$

$$+ (\alpha(u_{1,y} + u_x))_x \, dy$$

$$= f + \int_Y (\alpha u_{xx} w)_y$$

$$+ (\alpha(u_x w_y + u_x))_x \, dy$$

$$= f + u_{xx} \int_Y (\alpha w)_y + \alpha(w_y + 1) \, dy$$

$$= f + u_{xx} \int_Y \alpha_y w + 2\alpha w_y + \alpha \, dy$$

$$= f + u_{xx} \int_Y -(\alpha w_y)_y w + 2\alpha w_y + \alpha d$$

Lemma (ii)

$$= f + u_{xx} \int_Y \alpha w_y^2 + 2\alpha w_y + \alpha \, dy$$

$$= f + u_{xx} \int_Y \alpha (w_y + 1)(w_y + 1) \, dx$$

example:

$$\alpha(y) = (2 + \sin(2\pi y))^{-1}$$

$$\underline{u_\varepsilon}(x) = x(x-1) + \frac{\varepsilon}{2\pi}(x - 1/2)\left(1 - \cos\left(\frac{2\pi}{\varepsilon}x\right)\right) + \frac{\varepsilon^2}{(2\pi)^2}\left(\sin\left(\frac{2\pi}{\varepsilon}\right) - \frac{1}{2}\right)$$

$$\alpha^* = \int_0^1 \alpha\,(w_y + 1)^2\,dy$$

$$-(\alpha w_y)_y = \alpha_y$$

$$w(y) = \frac{1}{4\pi}(1 - \cos(2\pi y)) + \text{const}$$

$$w_y = \frac{1}{2}\sin(2\pi y)$$

$$\alpha^* = \int_0^1 \frac{(\frac{1}{2}\sin + 1)^2}{2 + \sin}\,dy$$

$$= \frac{1}{4}\int_0^1 \frac{(\sin + 2)^{\cancel{2}}}{\cancel{2 + \sin}} = \frac{2}{4} = \frac{1}{2}$$

homogenized solution

$$u_\varepsilon \longrightarrow u \quad \text{for } \varepsilon \to 0\ ?$$