Two-scale asymptotics

$$- \operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega \qquad u_\varepsilon\big|_{\partial\Omega} = 0$$

$$a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$$

$a : \mathbb{R}^d \to \mathbb{R}^d$ $\qquad Y = (0,1)^d$ - periodic

formal asymptotic expansion

ansatz:

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \ldots$$

where $u_i\left(x, \frac{x}{\varepsilon}\right)$ $\quad Y$-periodic

homogenized problem

for $u(x) = u_0\left(x, \frac{x}{\varepsilon}\right)$

$$- \operatorname{div}(a^* \nabla u) = f \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0$$

effective diffusion tensor

$$\alpha^* = (\alpha^*_{ij}) \in \mathbb{R}^{d \times d}$$

$$\alpha_{ij} = \int_Y \alpha \, (\nabla w_i + e_i)(\nabla w_j + e_j) \, dy$$

local cell problem

$$-\operatorname{div}\left(\alpha \left(\nabla w_i + e_i\right)\right) = 0 \quad \text{in } Y$$

w_i: Y-periodic

uniqueness up to constants

example:

$\Omega = (0,1)$, $\alpha(y) = \left(2 + \sin(2\pi y)\right)^{-1}$

$\alpha^* = 1/2$, $u(x) = x(x-1)$

$$\max_{x \in [0,1]} |u(x) - u_\varepsilon(x)| \to 0, \quad \varepsilon \to 0$$

in general ?

1.2.2. Weak convergence of sol.

weak convergence:

Hilbert space H, scalar prod. $(\cdot,\cdot)$

norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$

Definition:

$v_n \rightharpoonup v \iff (v_n, w) \to (v, w) \ \forall w \in H$

(i) uniqueness, linearity

$v_n \rightharpoonup v, \quad v_n \rightharpoonup v^*$

$\Leftrightarrow \quad (v_n, v - v^*) \to (v, v - v^*)$

$\to (v^*, v - v^*)$

$\Rightarrow (v, v - v^*) = (v^*, v - v^*)$

$\Rightarrow (v - v^*, v - v^*) = 0$

$\Rightarrow v = v^*$

(ii) (v_n) weakly convergent in H

$\Longrightarrow$ (v_n) is bounded in H

i.e. $\|v_n\| \leq C \quad \forall n \in \mathbb{N}$

(iii) $v_n \rightharpoonup v \iff$

$(v_n, \varphi) \to (v, \varphi) \quad \forall \varphi \in \mathcal{D} \subset H$ dense

(v_n) bounded

Proof: Banach-Steinhaus

$T_n : X \to Y$

$T_n(x) \to T(x) \quad \forall \varphi \in \mathcal{D} \subset X$

$\|T_n\| \leq C$

$\iff T_n(x) \to T(x) \quad \forall x \in X$

$T_n := (v_n, \cdot) : H \to \mathbb{R}$

(iv) (v_n) bounded in H

$\Rightarrow \exists v \in H : v_n \rightharpoonup v \quad n \in \mathbb{N}' \subset \mathbb{N}$

weak convergence of a subsequence

weak and strong convergence

(v) $v_n \to v \Rightarrow v_n \rightharpoonup H$

$|(v_n - v, w)| \le \|v_n - v_n\| \, \|w\| \to 0$

(vi) $v_n \to v$, $w_n \rightharpoonup w$

$(v_n, w_n) \Rightarrow (v, w)$

Proof: $|(v_n, w_n) - (v, w)| \le$

$\le |(v_n, w) - (v, w)| + |(v_n, w_n) - (v_n, w)|$
$\longrightarrow$

$\le \quad \dots \quad \|v_n\| \, \|w_n - w\| \to 0$
$\le C \qquad \longrightarrow 0$

attention:

$v_n \rightharpoonup v$ and $w_n \rightharpoonup w$

$\not\Rightarrow (v_n, w_n) \to (v, w)$

bounded linear mappings

Hilbert space L, $(\cdot,\cdot)_L$, $\|\cdot\|_L$

$T: H \to L$ linear, bounded

$\Rightarrow$ $v_n \rightharpoonup v$ in $H \Rightarrow Tv_n \rightharpoonup Tv$ in L

Proof: $(Tv_n, w)_L = (v_n, T^*w)_H$

$\to (v, T^*w)_H = (Tv, w)_L$

compact embedding

$H \subset L$ compact, i.e.

canonical injection $H \ni v \to Tv = v \in L$

is compact, i.e.

$U \subset H$ bounded $\Rightarrow \overline{T(U)}$ compact in L

i.e. $\sup\limits_{v \in U} \|v\| \le C$

$\Rightarrow (v_n) \subset U$ $\exists v \in L : v_n \to v$ in L
$n \in \mathbb{N}'$

(vii) $v_n \rightharpoonup v$ in H

$\Rightarrow$ $v_n \to v$ in L

proof:

a) $v_n \to v$ in $H \Rightarrow v_n \to v$ in L

T compact $\Rightarrow$ T bounded

T bounded: $Tv_n \to Tv$ in L

$\Leftrightarrow$ $v_n \to v$ in L

b) $v_n \rightharpoonup v$ in $H \Rightarrow (v_n)$ bounded in H

(v_n) compact in L

$\Rightarrow \exists v^* \in L$: $v_n \to v^*$ $n \in \mathbb{N}'$

$v_n \to v^*$ in L $\Rightarrow$ $v_n \rightharpoonup v^*$ in L

uniqueness of weak limit

$\Rightarrow$ $v = v^*$

holds for any subsequence $\mathbb{N}'$

$\Rightarrow$ convergence of the whole sequence

application to Sobolev spaces

$H = H_0^1(\Omega)$, $L = L^2(\Omega)$

attention:

$v_n \rightharpoonup v$ and $w_n \rightharpoonup w \not\Rightarrow v_n w_n \rightharpoonup v w$

in H or L

(see however div-curl lemma)

Rellich - Kondrachov Theorem

$H_0^1(\Omega) \subset L^2(\Omega)$ compact

see e.g. Adam Sobolev Spa

density of smooth function

$C_0^\infty(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega)$ dense

Lemma (example for weak conv.)

$\Omega \subset \mathbb{R}^d$ Lipschitz, bounded, Riemann measurable

$\psi \in L^2(Y)$, $Y = (0,1)^d$

extension $\psi : \mathbb{R}^d \to \mathbb{R}$ Y-periodic

$\psi_\varepsilon : \Omega \to \mathbb{R}$ $\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$

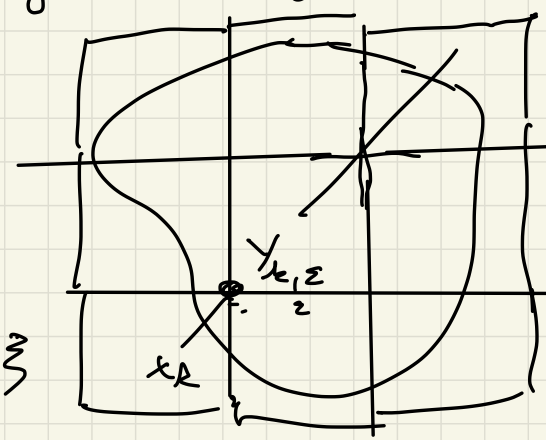
Then $\psi_\varepsilon \rightharpoonup \psi^* = \int_Y \psi(y)\,dy$ in $L^2(\Omega)$

Proof a) consistency $\mathcal{D} = C_0^\infty(\Omega)$!

$Y_{k,\varepsilon} = x_k + (0,\varepsilon)^d$

$x_k = \varepsilon k$, $k \in \mathbb{Z}^d$

$I_\varepsilon = \{k \in \mathbb{Z}^d \mid Y_{k,\varepsilon} \subset \Omega\}$



$\varphi \in C_0^\infty(\Omega)$

Riemannian sum:

$$\sum_{k \in I_\varepsilon} \varphi(x_k)\, \varepsilon^d \longrightarrow \int_\Omega \varphi \, dx$$

$$\forall \varphi \in C_0^\infty(\Omega)$$

for $\varepsilon \to 0$

let $\varphi \in C_0^\infty(\Omega)$, $\varepsilon > 0$ suff. small

$$\int_\Omega u_\varepsilon \varphi \, dx = \sum_{k \in I_\varepsilon} \int_{Y_{k,\varepsilon}} u_\varepsilon \varphi \, dx$$

$$= \sum_{k \in I_\varepsilon} \varphi(x_k) \int_{Y_{k,\varepsilon}} u_\varepsilon \, dx$$

$$+ \underbrace{\sum_{k \in I_\varepsilon} \int_{Y_{k,\varepsilon}} (\varphi(x_k) - \varphi(x))\, u_\varepsilon(x) \, dx}_{\longrightarrow 0}$$

$$\overset{y = x_k + \varepsilon x}{=} \sum_{k \in I_\varepsilon} \varphi(x_k)\, \varepsilon^d \underbrace{\int_Y u \, dy}_{= u^*}$$

$$\longrightarrow u^* \int_\Omega \varphi \, dx = (u^*, \varphi)_{L^2(\Omega)}$$

b) $\|u_\varepsilon\|_{L^2(\Omega)} \leq C$

$$I_\varepsilon^* = \{ k \in \mathbb{Z} \mid Y_{k,\varepsilon} \cap \Omega \neq \emptyset \}$$

$$\int_\Omega |u_\varepsilon|^2 dx \leq \sum_{k \in I_\varepsilon^*} \int_{Y_{k,\varepsilon}} |u_\varepsilon|^2 dx$$

$$\overset{y = x_k + \varepsilon x}{=} \sum_{k \in I_\varepsilon^*} \varepsilon^d \int_Y |u|^2 dy$$

$$= \|u\|^2_{L^2(Y)} \sum_{k \in I_\varepsilon^*} \varepsilon^d$$

$$\leq C \|u\|^2_{L^2(Y)}$$

assertion follows from (iii)