oscillating coefficient:

$$-\operatorname{div}(\alpha_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \quad u_\varepsilon\big|_{\partial\Omega} = 0$$

$$\alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right), \quad \alpha : \mathbb{R}^d \to \mathbb{R}^d$$

$$0 < \alpha_0 \le \alpha(y) \le \alpha_1, \quad Y = (0,1)^d\text{-periodic}$$

homogenized problem:

$$-\operatorname{div}(\alpha^* \nabla u) = f \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0$$

effective diffusion tensor:

$$\alpha^* = (\alpha^*_{ij}) \in \mathbb{R}^{d \times d}$$

$$\alpha_{ij} = \int_Y \alpha (\nabla w_i + e_i)(\nabla w_j + e_j)\,dy$$

local cell problems:

$$-\operatorname{div}(\alpha(\nabla w_i + e_i)) = 0 \text{ in } Y$$

w_i: Y-periodic

Lemma (example for weak conv.)

$\Omega \subset \mathbb{R}^d$ Lipschitz, bounded, Riemann measurable

$\psi \in L^2(Y)$, $Y = (0,1)^d$

Y-periodic extension: $\psi : \mathbb{R}^d \to \mathbb{R}$

$\psi_\varepsilon : \Omega \to \mathbb{R}$ $\quad \psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$

Then $\psi_\varepsilon \rightharpoonup \psi^* = \int_Y \psi(y)\, dy$ in $L^2(\Omega)$, for $\varepsilon \to 0$

i.e. $\int_\Omega \psi_\varepsilon v \, dx \to \int_\Omega \psi^* v \, dx \quad \forall v \in L^2(\Omega)$

Lemma (example for weak conv.)

$\Omega \subset \mathbb{R}^d$ Lipschitz, bounded, Riemann measurable

$\psi \in L^2(Y)$, $Y = (0,1)^d$

Y-periodic extension: $\psi : \mathbb{R}^d \to \mathbb{R}$

$\psi_\varepsilon : \Omega \to \mathbb{R}$ $\quad \psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$

Then $\psi_\varepsilon \rightharpoonup \psi^* = \int_Y \psi(y)\,dy$ in $L^2(\Omega)$,

i.e. $\int_\Omega \psi_\varepsilon v\,dx \to \int_\Omega \psi^* v\,dx \quad \forall v \in L^2(\Omega)$

Lemma:

$$\alpha_{ij} = \int_Y \alpha(y)(\nabla w_j + e_j)\cdot e_i\,dy$$
$$= \int_Y \alpha(y)(\nabla w_i + e_i)(\nabla w_j + e_j)\,dy$$

Proof: Exercise

Theorem

$\alpha \in C(\overline{Y}) \cap C^1(Y)$

($\Leftrightarrow$ classical sol. of cell problems)

Then: $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ for $\varepsilon \to 0$

Proof:

concept of proof

1) a priori estimate $\|u_\varepsilon\|_1 \leq$ const $\forall \varepsilon$

2) weak convergence of subsequence

$\exists u^*$: $u_{\varepsilon'} \rightharpoonup u^*$ in $H^1(\Omega)$

3) u^* is a solution of hom. prob. (*)

4) uniqueness of sol. u. of (*)

$\Leftrightarrow u^* = u$ and $u_\varepsilon \rightharpoonup u$

1. Step: Lax - Milgram

$$\| u_\varepsilon \|_1 \leq \frac{1}{\alpha_0} \| f \|_0 \quad \forall \varepsilon > 0$$

2. Step:

$\exists\, u^* \in H_0^1(\Omega)$: $\quad u_{\varepsilon'} \rightharpoonup u^*, \ \varepsilon \to 0$

$$\Leftrightarrow \quad u_{\varepsilon'} \rightharpoonup u^* \text{ in } L^2(\Omega)$$
$$\nabla u_{\varepsilon'} \rightharpoonup \nabla u^* \text{ in } L^2(\Omega)$$

compact embedding $H^1(\Omega) \subset L^2(\Omega)$

$$\Leftrightarrow \quad u_{\varepsilon'} \longrightarrow u^* \text{ in } L^2(\Omega)$$

Step 3: u^* solves hom. problem

oscillating test function

$$v_\varepsilon = \varphi + \varepsilon \sum_{i=1}^{d} \varphi_{x_i} w_{i,\varepsilon}$$

$\varphi \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ dense

$$w_{i,\varepsilon}(x) = w_i\left(\frac{x}{\varepsilon}\right) = w_i(y), \; y = \frac{x}{\varepsilon}$$

$$v_\varepsilon \in H_0^1(\Omega)$$

$$\| \varphi_{x_i} w_{i,\varepsilon} \|_0 \leq \text{const.} \quad \forall \varepsilon > 0$$

$$v_\varepsilon \longrightarrow \varphi \quad \text{in } L^2(\Omega), \; \varepsilon \longrightarrow 0$$

$$\nabla v_\varepsilon = \nabla \varphi + \sum_{i=1}^{d} \varepsilon \nabla \varphi_{x_i} w_{i,\varepsilon} + \varphi_{x_i} \nabla_y w_{i,\varepsilon}$$

$$= \sum_{i=1}^{d} \varphi_{x_i} (e_i + \nabla_y w_{i,\varepsilon}) + \varepsilon \sum_{i=1}^{d} \nabla \varphi_{x_i} w_{i,\varepsilon}$$

$$\| \nabla v_\varepsilon \|_0 \leq \text{const.}$$

test oscillating coefficient problem wrt v_ε

$$(f, v_\varepsilon) = (\alpha_\varepsilon \nabla u_\varepsilon, \nabla v_\varepsilon)$$

$$= \Big(\alpha_\varepsilon \nabla u_\varepsilon, \sum_{i=1}^{d} \varphi_{x_i} (e_i + \nabla_y w_{i,\varepsilon})\Big) + \delta(\varepsilon)$$

$$= -\Big(u_\varepsilon, \sum_{i=1}^{d} \mathrm{div}\big(\varphi_{x_i} \alpha_\varepsilon (e_i + \nabla_y w_{i,\varepsilon})\big)\Big) + \delta$$

$$= -\Big(u_\varepsilon, \sum_{i=1}^{d} \nabla \varphi_{x_i} \cdot \underbrace{\alpha_\varepsilon (e_i + \nabla_y w_{i,\varepsilon})}_{\psi_{\varepsilon,i}}\Big)$$

$$-\Big(u_\varepsilon, \sum_{i=1}^{d} \varphi_{x_i} \underbrace{\mathrm{div}\big(\alpha_\varepsilon (e_i + \nabla_y w_{i,\varepsilon})\big)}_{=0}\Big) + \delta$$

$$\psi_{\varepsilon,i} = \alpha_\varepsilon (e_i + \nabla_y w_{i,\varepsilon}) \xrightarrow{\text{in } L^2} \underbrace{\int_Y \alpha (e_i + \nabla_y w_i)\, dy}_{F_i \in \mathbb{R}^d}$$

$$F_i \cdot e_e = \alpha^*_{ei}$$

passing to the limit $\varepsilon \to 0$:

$$(f, \varphi) = -\left(u^*, \sum_{i=1}^{d} \nabla \varphi_{x_i} \cdot F_i\right)$$

$$= -\left(u^*, \sum_{i=1}^{d} \sum_{\ell=1}^{d} \varphi_{x_i x_\ell} \underbrace{F_i \cdot e_\ell}_{\alpha^*_{\ell i}}\right)$$

$$= -\left(u^*, \sum_{\ell=1}^{d} \left(\sum_{i=1}^{d} \alpha^*_{\ell i} \varphi_{x_i}\right)_{x_\ell}\right)$$

$$= -\left(u^*, \operatorname{div}(\alpha^* \nabla \varphi)\right)$$

$$= (\nabla u^*, \alpha^* \nabla \varphi)$$

$$\overset{\alpha^* \text{ sym.}}{=} (\alpha^* \nabla u^*, \nabla \varphi)$$

$\varphi \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ dense

$\Leftrightarrow$ u^* solves hom. problem.

Step 4: α^* s.p.d exercise

$\Leftrightarrow$ uniqueness of sol. of h.p.

$\Leftrightarrow$ $u_\varepsilon \to u$ $\quad \varepsilon \to 0$ $\quad$ ☐

further comments:

a) strong regularity assumptions on α <-> classical sol. of cell.p

generalization: weak sol.

$$w_i \in H^1_{per}(Y) : (\alpha(\nabla w_i + e_i), \nabla v) = 0 \quad \forall v \in H^1_{per.}(Y)$$

two-scale convergence (Allaire 92)

manuscript Ben Schweizer

b) corrector results

(Moscow & Vogelius 97)

$$u_\varepsilon - (\underline{u} + \varepsilon u_1) \longrightarrow 0 \text{ in } H^1(\Omega)$$

remember:

$$u_1(x, \frac{x}{\varepsilon}) = \sum_{i=1}^{d} u_{x_i}(x)\, w_i(\frac{x}{\varepsilon})$$

where w_i : $\int_Y w_i = 0$

1.2.3 Homogenization of porous media flow

conservation of mass

$$\varrho_t = \operatorname{div}(\varrho v) \qquad \varrho = \text{const}$$

$$\Leftrightarrow \operatorname{div} u = 0 \qquad u \text{ velocity}$$

conservation of momentum

$$u_t + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f$$

pressure p

low Reynoldsnumber : $(u \cdot \nabla u) \approx 0$

stationary flow : $u_t \approx 0$

Stokes equation

$$\nabla p - \nu \Delta u = f \qquad \operatorname{div} u = 0$$

$$u = 0 \text{ on } \partial\Omega$$

weak formulation $f \in L^2(\Omega)^d$

find $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$:

$$(\nu \nabla u, \nabla v) - (p, \operatorname{div} v) = (f, v) \qquad \forall v \in H_0^1(\Omega)^d$$

$$-(\operatorname{div} u, q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

saddle point problem

Ladyzhenskaya - Babuska - Brezzi

$$\inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)^d} \frac{(\operatorname{div} v, q)}{\|v\|_{H_0^1(\Omega)^d} \|q\|_{L^2(\Omega)}} \geq \alpha_0 > 0$$

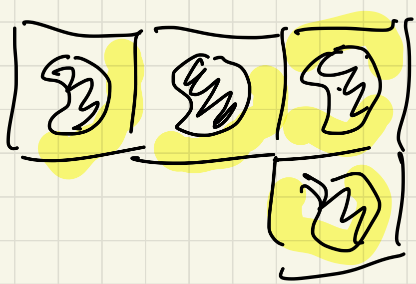
Braess Chapter $\overline{\underline{III}}$ § 5

Stokes flow through porous medium

$$\left.\begin{aligned} \nabla p_\varepsilon - \varepsilon^2 \Delta u_\varepsilon &= f \\ \operatorname{div} u_\varepsilon &= 0 \end{aligned}\right\} \text{ in } \Omega_\varepsilon$$

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{k \in I_\varepsilon} Y^\circ_{k,\varepsilon}$$

$$Y^\circ_{k\varepsilon} = x_k + \varepsilon Y^\circ \qquad Y^\circ \subset Y \text{ solid part}$$



existence & uniqueness of $(u_\varepsilon, p_\varepsilon)$

extension from Ω_ε to Ω by 0

Theorem

Consider the hom. problem

$$\left.\begin{aligned} u &= \alpha^*(f - \nabla p) \\ \operatorname{div} u &= 0 \end{aligned}\right\} \Omega$$

$$\Leftrightarrow \quad -\operatorname{div}(\alpha^* \nabla p) = -\operatorname{div}(\alpha^* f)$$

$$u \cdot n = 0 \quad \text{on } \partial\Omega$$

$$\frac{\partial}{\partial n} p = \alpha^* f \cdot n \quad \text{on } \partial\Omega$$

with

$$\alpha^* = (\alpha^*_{ij})$$

$$\alpha^*_{ij} = \int_{Y \setminus Y^\circ} \nabla w_i \nabla w_j \, dy$$

cell Stokes problem

$$\nabla q_i - \Delta w_i = e_i \quad \operatorname{div} w_i = 0$$

in $Y \setminus Y^\circ$ $\quad w_i = 0 \; \partial Y$

q_i, w_i $\quad Y$ periodic

$(u_\varepsilon, p_\varepsilon) \longrightarrow (u, p)$

in $L^2(\Omega)^d \times H^1(\Omega)/\mathbb{R}$

Allaire (1989)

$\Leftrightarrow$ website