1.3 Multiscale finite elements

1.3.1 Subspace correction

H Hilbert space, $(\cdot,\cdot)_H$, $\|\cdot\|_H = (\cdot,\cdot)_H^{1/2}$

example: $H = H_0^1(\Omega)$

variational equalities

$$u \in H : a(u,v) = \ell(v) \quad \forall v \in H$$

bilinear form $a(\cdot,\cdot)$

symmetric, elliptic, i.e.

$$\gamma \|v\|_H^2 \le a(v,v)$$

$$|a(v,w)| \le \Gamma \|v\|_H \|w\|_H$$

energy scalar product $a(\cdot,\cdot)$

energy norm $\|\cdot\| = a(\cdot,\cdot)^{1/2}$

norm equivalence

$$\mu^{1/2} \| v \|_H \le \| v \| \le T^{1/2} \| v \|_H$$

spectral property of elliptic bilinear forms

Hilbert space L, $(\cdot , \cdot)_L$, $\| \cdot \|_L$

$H \subset L$ compact embedding
dense

$L = L^2(\Omega)$

Definition

μ eigenvalue of $a(\cdot , \cdot)$
with eigenfunction $e \in H$, iff

$$a(e, v) = \mu \, (e, v)_L \quad \forall v \in H$$

operator notation

$$a(e, v) = (Ae, v)_L = \mu (e, v)_L$$

$$Ae = \mu e \quad \text{in } L$$

Theorem 1

Let $\dim H = \infty$. Then there are a-orthogonal eigenfunctions e_i, $i \in \mathbb{N}$, with eigenvalues μ_i, $i \in \mathbb{N}$ such that

$$0 < \mu_1 \leq \mu_2 \leq \ldots$$

and $\mu_i \to \infty$ for $i \to \infty$

Proof: Raviart & Thomas

Introduction à l'analyse numérique.

example: $\Omega = (0,1)$

$H = H_0^1(\Omega)$, $L = L(\Omega)$

$a(v,w) = (v', w')$, $A = -\frac{d^2}{dx^2}$

$e_i = \sin(i\pi x)$, $\mu_i = (i\pi)^2$

geometric interpretation

a - orthogonal eigenfunctions

$\Longleftrightarrow$

scale of frequencies

"almost orthogonal" functions

Ritz - Galerkin approximation

$S_N \subset H$ finite dim. N

canonical isomorphism

$$S_N \ni v \longrightarrow i_N(v) = \underline{v} \in \mathbb{R}^N$$

representation of $a(\cdot,\cdot)|_{S_N \times S_N}$

$$a(v,w) = \langle A_N \underline{v}, \underline{w} \rangle_{\mathbb{R}^N}, \; v,w \in S_N$$

$$\kappa_2(A_N) = \frac{\mu_{max}(A_N)}{\mu_{min}(A_N)} \longrightarrow \infty$$

example: $\Omega = (0,1)$, $a(v,w) = (v', w')$

$T_h = \{ t = [x_{i-1}, x_i] \mid i = 1, \dots, N+1 \}$

$x_i = i h$, $h = 1/N+1$

$S_N = \{ v \in H_0^1(\Omega) \mid v|_t \text{ affin } \forall t \in T_h \}$

basis: $\lambda_i \in S_N$ $\quad \lambda_i(x_j) = \delta_{ij}$

$i_N(v) = (v(x_i))_{i=1}^N = \underline{v} \in \mathbb{R}^N$

stiffness matrix:

$A_N = (a_{ij})_{i,j=1}^N \quad a_{ij} = a(\lambda_i, \lambda_j)$

$a(v, w) = \langle A \underline{v}, \underline{w} \rangle_{\mathbb{R}^N}$

$\kappa(A_N) = O(N^2)$

subspace decomposition

subspaces $V_i \subseteq H$, $i = 1, \dots, n$

$$H = V_1 + \dots + V_n$$

Ritz projections: $P_i : H \to V_i$

$P_i w \in V_i$:

$$a(P_i w, v) = a(w, v) \quad \forall v \in V_i$$

preconditioner:

$$T = P_1 + \dots + P_n$$

Jacobi-type preconditioner

example: $H = S_N$

$$V_i = \operatorname{span}\{\lambda_i\}$$

$$a(v, w) = \langle A_N \underline{v}, \underline{w} \rangle_{\mathbb{R}^N}$$

$$P_i w = w_i \lambda_i, \quad w_i \in \mathbb{R}$$

$$a(w_i \lambda_i, \lambda_i) = a(w, \lambda_i)$$

$$w_i = a(\lambda_i, \lambda_i)^{-1} a(w, \lambda_i)$$

$$= \frac{1}{a_{ii}} (A_N \underline{w})_i$$

$$Tw = \sum_{i=1}^{n} w_i \lambda_i$$

$$\underline{Tw} = D^{-1} A_N \underline{w}$$

$$D = \operatorname{diag}(A_N)$$

Jacobi preconditioner

Lemma 2

$$a(Tv, w) = a(v, Tw), \quad v, w \in H$$

Proof:

$$a(P_i v, w) = a(v, P_i w)$$

because $P_i : H \to V_i$ is a-orthogonal

Lemma 3

$$\| I - \omega T \| = \sup_{v \neq 0} \frac{\| (I - \omega T) v \|}{\| v \|}$$

$$= \sup_{\mu \in \sigma(T)} | 1 - \omega \mu |$$

Spectrum $\sigma(T) = \{ \mu \in \mathbb{R} \mid (\mu I - T)^{-1} \in B(H) \}$

Proof: $T : H \to L$ compact

a-orthogonal basis of eigenfunct e_i

$v = \sum v_i e_i$, $Tv = \sum \mu_i v_i e_i$ in L

preconditioned Richardson iteration

$$u_{\nu+1} = u_\nu + \omega T(u - u_\nu)$$

in H

$$\Leftrightarrow \quad a(u_{\nu+1}, v) = a(u_\nu, v) + \omega\, a(T(u-u_\nu), v)$$

$$\Leftrightarrow \quad A u_{\nu+1} = A u_\nu + \omega A T (u - u_\nu)$$

desired $T \approx I$

goal

$$1/K_1\, a(v,v) \leq a(Tv, v) \leq K_2\, a(v,v)$$

$$\Rightarrow \quad \kappa(T) \leq K_1 K_2$$

example: damped Jacobi method

evaluation of $u_{\nu+1}$:

$$T(u-u_\nu) = \sum_{i=1}^{n} w_i \quad , \quad w_i = P_i(u-u_\nu)$$

$w_i \in V_i$:

$$a(w_i, v) = a(u-u_\nu, v)$$

$$= \underbrace{l(v) - a(u_\nu, v)}_{\text{residual}}$$

$$\forall v \in V_i$$

n "smaller" problems

instead of the given one

Proposition 4

Let $\sigma(T) = [\mu_{min}, \mu_{max}]$

with $0 < \mu_{min} < \mu_{max} < \infty$

and $\omega \in (0, 2/\mu_{max})$

Then $(u_\nu)_{\nu \in \mathbb{N}} \longrightarrow u$ in H

for all $u_0 \in H$.

The optimal damping param.

$\omega = 2/(\mu_{max} + \mu_{min}) = 2/\mu_{min}(1 + \gamma_\Sigma(T))$

with $\gamma_\Sigma(T) = \frac{\mu_{max}}{\mu_{min}}$ leads to

$$\varsigma_{opt} = \frac{\gamma_\Sigma(T) - 1}{\gamma_\Sigma(T) + 1}$$

Proof:

a) contraction

$$\| v - \omega T(u-v) - (w - \omega T(u-w)) \|$$

$$\leq \| I - \omega T \| \, \| v - w) \|$$

$$\leq \max_{\mu \in \sigma(T)} | 1 - \omega \mu | \, \| v - w \|$$

$$\varrho(\omega) = \max_{\mu \in \sigma(T)} | 1 - \omega \mu |$$

b) $\varrho(\omega) < 1$ for $\omega \in (0, 2/\mu_{max})$

$$1 - \omega \mu < 1 \quad \text{as } \mu > 0 \ \forall \mu \in \sigma(T)$$

$$\omega \mu - 1 < 1 \Rightarrow \omega < \frac{2}{\mu} \ \forall \mu \in \sigma(T)$$

c) $$\min_{\omega} \varrho(\omega) = \min_{\omega} \max_{\mu} | 1 - \omega \mu |$$

$$= \min_{\omega} \max \{ |1 - \omega \mu_{min}|, |1 - \omega \mu_{max}| \}$$

$$\Leftrightarrow 1 - \omega_{opt} \mu_{min} = \mu_{opt} \mu_{max} - 1$$

main assumption on splitting

(V1) For all $v \in H$ there is

(*) $\quad v = v_1 + \ldots + v_n \,, \; v_i \in V_i$

such that

$$\sum_{i=1}^{n} \| v_i \|^2 \leq K_1 \, \| v \|^2$$

(V2) Every decomposition of the form (*) satisfies

$$\| v \|^2 \leq K_2 \sum_{i=1}^{n} \| v_i \|^2$$

example:

Let V_i be pairwis a-orthogonal

Then (V1), (V2) hold with

$K_1 = K_2 = 1$

Proof: $v = \sum_{i=1}^{n} v_i \quad , v_i \in V_i$

$$\|v\|^2 = a\left(\sum_{i=1}^{n} v_i , \sum_{j=1}^{n} v_j\right)$$

$$= \sum_{i,j=1}^{n} a(v_i , v_j)$$

$$= \sum_{i=1}^{n} a(v_i ; v_i) = \sum_{i=1}^{n} \|v_i\|^2$$