Subspace correction

$$u \in H: \quad a(u,v) = \ell(v) \quad \forall v \in H$$

$$\|\cdot\| = a(\cdot,\cdot)^{1/2}$$

$$H = V_1 + \ldots + V_n$$

Ritz projections $P_i : H \to V_i$

$$P_i w \in V_i : \quad a(P_i w, v) = a(w, v) \quad \forall v \in V_i$$

Jacobi-type preconditioner

$$T = P_1 + \ldots + P_n : H \to H$$

preconditioned Richardson iteration

$$u_{\nu+1} = u_\nu + \omega T(u - u_\nu) \quad \nu \in \mathbb{N}$$

convergence (Proposition 3)

$$\sigma(T) = [\mu_{min}, \mu_{max}]$$

$$\| u - u^{\nu+1} \| \leq \varrho(\omega) \| u - u^{\nu} \|$$

$$\varrho(\omega) = \max_{\mu \in \sigma(T)} | 1 - \omega \mu | < 1$$

$$\text{for } 0 < \omega < \frac{2}{\mu_{max}}$$

$$\varrho_{opt} = \frac{\kappa(T) - 1}{\kappa(T) + 1} = 1 - \frac{2}{\kappa(T) + 1} < 1$$

$$\omega_{opt} = \frac{2}{\mu_{min} + \mu_{max}}$$

$$\kappa(T) = \frac{\mu_{max}}{\mu_{min}}$$

main assumptions on splitting

(V1) $\forall v \in H \ \exists \ v = v_1 + \dots + v_n$

with $v_i \in V_i$ such that

$$\sum_{i=1}^{n} \| v_i \|^2 \leq K_1 \| v \|^2$$

(V2) $\forall v \in H \ \forall v = v_1 + \dots + v_n$

with $v_i \in V_i$ satisfy

$$\| v \|^2 \leq K_2 \sum_{i=1}^{n} \| v_i \|^2$$

with $K_1, K_2 > 0$

Quantification of a-orthogonality

Lemma 4

Assume that (V1) and (V2) hold

Then $T = P_1 + \ldots + P_n$

satisfies

$$\frac{1}{K_1} a(v,v) \leq a(Tv,v) \leq K_2 a(v,v)$$

for all $v \in H$

consequence for $\dim H < \infty$

$$K_2 \geq \sup_{v \neq 0} \frac{a(Tv,v)}{a(v,v)} = \mu_{max}(T)$$

$$\geq \mu_{min}(T) = \inf_{v \neq 0} \frac{a(Tv,v)}{a(v,v)} \geq \frac{1}{K_1}$$

$\sigma(T) \subset [1/K_1, K_2]$, $\quad \kappa(T) \leq K_1 K_2$

$\Leftrightarrow$ convergence for $0 < \omega < \frac{2}{K_2} \leq \frac{2}{\mu_{max}}$

$$\varrho_{opt} = 1 - \frac{2}{\kappa(T)+1} \leq 1 - \frac{2}{K_1 K_2 + 1}$$

Proof:

a) Let $v \in H$ and $v = v_1 + \ldots + v_n$ as in (V1)

$$\|v\|^2 = a(v, v) = \sum_{i=1}^{n} (v_i, v)$$

$$= \sum_{i=1}^{n} a(P_i v_i, v) = \sum_{i=1}^{n} a(v_i, P_i v)$$

$$\overset{C.S.}{\leq} \sum_{i=1}^{n} \|v_i\| \, \|P_i v\|$$

$$\leq \left(\sum_{i=1}^{n} \|v_i\|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \|P_i v\|^2 \right)^{1/2}$$

$$\overset{(V_1)}{\leq} K_1^{1/2} \|v\| \left(\sum_{i=1}^{n} a(P_i v, P_i v) \right)^{1/2}$$

$$= K_1^{1/2} \|v\| \left(\sum_{i=1}^{n} a(P_i v, v) \right)^{1/2}$$

$$= K_1^{1/2} \|v\| \, a(Tv, v)^{1/2}$$

$\leftrightarrow$ lower bound

b) upper bound, $v \in H$

$$a(Tv, v) \leq \|Tv\| \|v\|$$

$$= \left\| \sum_{i=1}^{n} P_i v \right\| \|v\|$$

$$\overset{(V2)}{\leq} K_2^{1/2} \left(\sum_{i=1}^{n} \|P_i v\|^2 \right)^{1/2} \|v\|$$

$$= K_2^{1/2} \left(\sum_{i=1}^{n} a(P_i v, v) \right)^{1/2} \|v\|$$

$$= K_2^{1/2} \; a(Tv, v)^{1/2} \|v\|$$

$\Leftrightarrow$ upper bound

Theorem 5

Let (V1) and (V2) hold.
Then $\sigma(T) \subset [1/K_1, K_2]$
and thus $\kappa(T) \leq K_1 K_2$

Proof:
We show $\mu \notin [1/K_1, K_2] \Rightarrow \mu \notin \sigma(T)$

a) $\mu < 1/K_1$ Then

$$(v, w)_\mu := a((T - \mu I)v, w)$$

is a scalar product on H

$$(v, v)_\mu = a((T - \mu I)v, v)$$

$$= a(Tv, v) - \mu\, a(v, v)$$

$$\geq \underbrace{\left(\frac{1}{K_1} - \mu\right)}_{>0} a(v, v)$$

$$\Leftrightarrow \quad \|v\|_\mu \geq (1/K_1 - \mu)^{1/2} \|v\|$$

Let $w \in H$ and consider

$\cdot\ u \in H: \quad (u, v)_\mu = a(w, v) \quad \forall v \in H$

(i) Riesz: existence & uniqueness of u

(ii) $\cdot\ \| u \|_\mu \leq (1/\kappa_1 - \mu)^{-1/2} \| w \|$

$$\| u \|_\mu^2 = (u, u)_\mu = a(w, v)$$
$$\leq \|w\| \|u\| \leq \| w \| (1/\kappa_1 - \mu)^{-1/2} \|u\|_\mu$$

(i) $\iff \quad a((T - \mu I) u, v) = a(w, v)$

$\iff \quad (T - \mu I) u = w$

$\iff \quad u := (T - \mu I)^{-1} w$

(ii) $\quad \| w \| \geq (1/\kappa_1 - \mu)^{1/2} \| u \|_\mu$

$$\geq (1/\kappa_1 - \mu) \| u \|$$

$$= (1/\kappa_1 - \mu) \| (T - \mu I)^{-1} w \|$$

b) Let $\mu > K_2$

Then $(\mu I - T)^{-1} \in B(H)$ follows in the same way. $\boxed{}$

Corollary 6

Let (V1) and (V2) hold.

Then the precond. Richardson iteration converges for

$0 < \omega < 2/K_2 \leq \frac{2}{\mu_{max}}$ and for the "optimal" choice $\omega^* = \frac{2}{1/K_1 + K_2}$

with convergence rate

$$\varrho_{opt} \leq \varrho(\omega^*) = 1 - \frac{2}{K_1 K_2 + 1} \quad \text{and } \kappa(T) \leq K_1 K_2$$

Proof.: obvious

application

Let $V \subset H$ be a closed subspace and $C : H \to V$ the Ritz projection defined by

$$Cw \in V : \quad a(Cw, v) = a(w, v) \quad \forall v \in V$$

Let the decomposition

$$V = V_1 + \ldots + V_n$$

satisfy (V1) and (V2) with H replaced by V and

$$T = P_1 + \ldots + P_n : H \to V$$

Corollary 7

The preconditioned Richardson

$$C_{\nu+1} v = C_\nu v + \omega T (v - C_\nu v)$$

with $C_0 v = 0$ and

$$\omega = \frac{2 \kappa_1}{\kappa_1 \kappa_2 + 1}$$

converges according to

$$\| C v - C_\nu v \| \le \varrho^\nu \| C v \| , \quad v \in H$$

with

$$\varrho = 1 - \frac{2}{\kappa_1 \kappa_2 + 1}$$

Proof: a) $C - C_\nu = (I - \omega T)^\nu C$

induction:

$\nu = 0$: $(I - \omega T)^0 C = C = C - C_0$

because $C_0 = 0$

$\nu \to \nu + 1$:

$$C - C_{\nu+1} = C - C_\nu - \omega T (I - C_\nu)$$

$$= (I - \omega T)^\nu C - \omega T (I - C - (I - \omega T)^\nu C)$$

$$= (I - \omega T)^{\nu+1} C - \omega \underbrace{T (I - C)}_{=0}$$

$(I - C) H$ is orthogonal to V and thus to V_i

$\Leftrightarrow P_i (I - C) = 0 \quad i = 1, \ldots, n$

b) $\| C v - C_\nu v \| = \| (I - \omega T)^\nu C v \|$

$\leq \| (I - \omega T) |_V \|^\nu \| C v \|$

$\leq \rho(\omega)^\nu \| C v \|$