1.3.2 Multiscale finite elements

(Hou & Wu 97, Hou & Efendiev 09,
Målqvist & Peterseim (LOD) 14,
...., Kh, Peterseim, Yserentant 18)

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, polygonal
$A : \Omega \to \mathbb{R}^{d \times d}$, symmetric, and

$$\delta |\eta|^2 \le \eta \cdot A(x)\eta \le M |\eta|^2 \quad \forall \eta \in \mathbb{R}^d \quad \text{a.e. } \Omega$$

and $0 < \delta \le M \in \mathbb{R}$

Remark: Special case:

$$A(x) = \alpha\left(x, \frac{x}{\varepsilon}\right) I$$

no scale separation,
no periodicity

bilinear form

$$a(v, w) = \int_\Omega \nabla v \cdot A \nabla w \, dx$$

$H = H_0^1(\Omega)$, $\ell(v) = \int_\Omega f v \, dx$

multiscale problem

$u \in H$: $a(u, v) = \ell(v) \;\; \forall v \in H$

ellipticity:

$$\gamma \|v\|_1^2 \leq a(v, v)$$

$$|a(v, w)| \leq \Gamma \|v\|_1 \|w\|_1$$

$\Leftrightarrow$ existence, uniqueness, stability of solutions

condition number Γ/γ

depends on global contrast M/δ

classical finite elements

Triangulation $\mathcal{T}$: $\overline{\Omega} = \bigcup_{t \in \mathcal{T}} t$

interior vertices : $\mathcal{N}$

shape regularity

$$r_t / \rho_t \leq \text{const} \quad \forall t \in \mathcal{T}$$

local mesh size

$H : \Omega \to \mathbb{R}$ $\quad H(x) = \operatorname{diam} t,$

$x \in \operatorname{int} t$

$H \in L^\infty(\Omega)$

no quasi-uniformity of $\mathcal{T}$

local contrast M_t / δ_t

$$\delta_t |\eta|^2 \leq \eta \cdot A(x)\eta \leq M_t |\eta|^2$$

$$\forall x \in t$$

classical finite element space

$$S_H = \text{span}\,\{ \lambda_p \mid p \in \mathcal{N} \}$$

$$u_H \in S_H : \quad a(u_H, v) = \ell(v) \quad \forall v \in S_H$$

error estimates for special case

$$\| u - u_H \|_1 \leq \frac{1}{\gamma}\, \text{const.}\, \| H \|_\infty / \varepsilon$$

desired: $\mathcal{W} \subset H$, $\dim \mathcal{W} = \dim S_H$

$$w \in \mathcal{W} : \quad a(w, v) = \ell(v) \quad \forall v \in \mathcal{W}$$

$$\| u - w \|_1 \leq C \, \| H f \|_0$$

with $C = C(\,{}^{M}\!/\!_{\gamma}, \text{const.})$

linear projection operator

from now on let

$$\Pi : H \longrightarrow S_H$$

denotes a linear projection

a) $\Pi(\alpha v + \beta w) = \alpha \Pi v + \beta \Pi w$

$$\Pi^2 = \Pi$$

with the properties

b) $|\Pi v|_1 \leq c_1 |v|_1$ (boundedness)

c) $\| H^{-1}(v - \Pi(v)) \|_0$

$$\leq c_2 |v|_1$$

(approximation property)

Proposition 8

Let $d = 1$ ($H_0^1(\Omega) \subset C(\bar{\Omega})$)

Then nodal interpolation

$$I_H v = \sum_{p \in N} v(p) \lambda_p$$

satisfies a), b), c)

Proof: a)

b), c) : local affine transformation

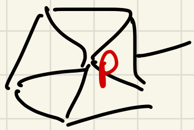
$\Leftrightarrow$ straightforward

Proposition 9 (Clément operator 75)

$$\Pi w = \sum_{p \in \mathcal{N}} w_p(p)\, \lambda_p \qquad w \in H^1_0(\Omega)$$

with $w_p \in S(\omega_p) = \{ v|_{\omega_p} \mid v \in S_H \}$

$\omega_p = \operatorname{supp} \lambda_p$



ω_p

$w_p \in S(\omega_p) : (w_p, v) = (w, v) \quad \forall v \in S(\omega_p)$

(local L^2-projection)

Π satisfies a), b), c).

Proof for the quasiuniform case:
exercise

Remark: a) $w \in S_H$, $w|_{\omega_p} \in S(\omega_p)$
$\Leftrightarrow w_p = w \Leftrightarrow \Pi w = w$

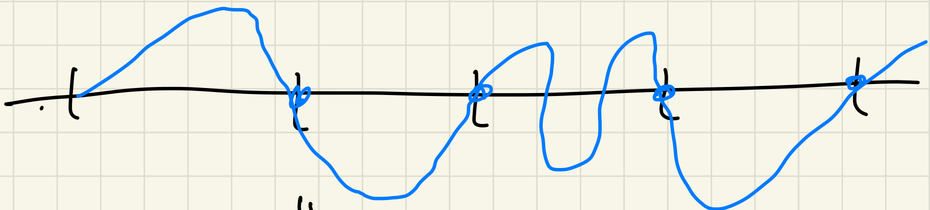
linearity follows from linearity of local L^2-projection

orthogonal decomposition of H

(Målqvist & Peterseim 14)

$$\mathcal{V} = \ker \Pi = \{ v \in H \mid \Pi v = 0 \}$$

example: $d = 1$, $\Pi = I_H$



$\mathcal{V}$ is "high-frequency"

a-orthogonal projection

$$C : H \longrightarrow \mathcal{V}$$

$$Cw \in \mathcal{V}: \quad a(Cw, v) = a(w, v) \quad \forall v \in H$$

a-orthogonal complement of $\mathcal{V}$

$$W := (I - C) H$$

Lemma 10

a) $(\underline{I} - C)v = (\underline{I} - C)\Pi v \quad \forall v \in H$

b) $\mathcal{W} = (\underline{I} - C) S_H$

c) $\dim \mathcal{W} = \dim S_H$

d) $\mathcal{W} = \operatorname{span}\{(\underline{I} - C)\chi_p \mid p \in \mathcal{N}\}$

Proof:

a) $(\underline{I} - C)v = (\underline{I} - C)(\Pi v + \underbrace{(\underline{I} - \Pi)v}_{\in V})$

$$= (\underline{I} - C)\Pi v + \cancelto{=0}{(\underline{I} - C)C(\underline{I} - \Pi)} v$$

b) $\mathcal{W} = (\underline{I} - C)H = (\underline{I} - C)\Pi H = (\underline{I} - C) S_H$

c) we show $\left((\underline{I} - C)|_{S_H}\right)^{-1} = \Pi|_{\mathcal{W}}$, i.e.

(i) $(\underline{I} - C)\Pi w = w \quad \forall w \in \mathcal{W}$

(ii) $\Pi(\underline{I} - C)v = v \quad \forall v \in S_H$

(i) $(I - C)\Pi w = w \quad \forall w \in W$

$w = (I - C)v \in W$ for some $v \in S_H$

$(I - C)\Pi w = (I - C)\Pi(I - C)v$

$= (I - C)(\Pi v - \Pi C v)$

$= (I - C) \; v = w$

(ii) $\Pi(I - C)v = v \quad \forall v \in S_H$

$\Pi(I - C)v = \Pi v - \Pi C v$

$= v$

d) $W = (I - C) S_H$

$= \{ w \mid (I - C) \sum_p v_p \lambda_p \}$

$= \{ w \mid \sum_p v_p (I - C) \lambda_p \}$

note: $\Pi (I - C)\lambda_p = \lambda_p$

modified finite element approx.

$w = P_{\mathcal{W}} u \in \mathcal{W}$:

$$a(w, v) = (f, v) \quad \forall v \in \mathcal{W}$$

Proposition 11

(Peterseim, Hughes & Sangalli)

a) $w = (\underline{I} - C)\, u$

b) $w = (\underline{I} - C)\, \Pi u$

Proof:

a) $a((\underline{I} - C) u, v) =$

$= a(u, v) - a(Cu, v)$

$= l(v) - \cancel{a(u, Cv)}^{=0}$

$= l(v) \quad \forall v \in \mathcal{W} = (\underline{I} - C) H$

b) Lemma 10 a) $\blacksquare$